THE AFFINE YOKONUMA–HECKE ALGEBRA AND THE PRO-p-IWAHORI–HECKE ALGEBRA

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Abstract. We prove that the affine Yokonuma–Hecke algebra defined by Chlouveraki and Poulain d'Andecy is a particular case of the pro-p-Iwahori–Hecke algebra defined by Vignéras.

1. Introduction

A family of complex algebras $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$, called affine Yokonuma–Hecke algebras, has been defined and studied by Chlouveraki and Poulain d'Andecy in [ChPo2]. The existence of these algebras has been first mentioned by Juyumaya and Lambropoulou in [JuLa1]. These algebras, which generalise both affine Hecke algebras of type A and Yokonuma–Hecke algebras [Yo], are used to determine the representations of Yokonuma–Hecke algebras [ChPo1] and construct invariants for framed and classical knots in the solid torus [ChPo2]. Moreover, when q^2 is a power of a prime number p and $d = q^2 - 1$, one can verify that the affine Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ is isomorphic to the convolution algebra of complex valued and compactly supported functions on the group $\mathrm{GL}_n(F)$, with F a suitable p-adic field, that are bi-invariant under the pro-p-radical of an Iwahori subgroup (cf. [Vi1]). It is natural to ask whether there is a family of algebras that generalises, in a similar way, affine Hecke algebras in arbitrary type.

In a recent series of papers [Vi2, Vi3, Vi4], Vignéras introduced and studied a large family of algebras, called pro-p-Iwahori–Hecke algebras. They generalise convolution algebras of compactly supported functions on a p-adic connected reductive group that are bi-invariant under the pro-p-radical of an Iwahori subgroup, which play an important role in the p-modular representation theory of p-adic reductive groups (see [AHHV] for instance).

In this note, we show that the algebra $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ of Chlouveraki and Poulain d'Andecy is a pro-p-Iwahori–Hecke algebra in the sense of Vignéras [Vi2].

2. The affine Yokonuma–Hecke algebra

Let $d, n \in \mathbb{Z}_{>0}$. Let q be an indeterminate or a non-zero complex number, and set $\mathcal{R} := \mathbb{C}[q, q^{-1}]$. We denote by $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ the associative algebra over $\mathcal{R}$ generated by elements

$$t_1, \ldots, t_n, g_1, \ldots, g_{n-1}, X_1, X_1^{-1}$$

subject to the following defining relations:

$$(2.1)\quad \begin{array}{llcll}
\text{(br1)} & g_i g_j & = & g_j g_i & \text{for all } i,j = 1,\ldots,n-1 \text{ such that } |i-j|>1,\\
\text{(br2)} & g_i g_{i+1} g_i & = & g_{i+1} g_i g_{i+1} & \text{for all } i = 1,\ldots,n-2,\\
\text{(fr1)} & t_i t_j & = & t_j t_i & \text{for all } i,j = 1,\ldots,n,\\
\text{(fr2)} & g_i t_j & = & t_{s_i(j)} g_i & \text{for all } i = 1,\ldots,n-1 \text{ and } j = 1,\ldots,n,\\
\text{(fr3)} & t_j^d & = & 1 & \text{for all } j = 1,\ldots,n,\\
\text{(aff1)} & X_1\, g_1 X_1 g_1 & = & g_1 X_1 g_1\, X_1 & \\
\text{(aff2)} & X_1 g_i & = & g_i X_1 & \text{for all } i = 2,\ldots,n-1,\\
\text{(aff3)} & X_1 t_j & = & t_j X_1 & \text{for all } j = 1,\ldots,n.\\
\text{(quad)} & g_i^2 & = & 1 + (q-q^{-1})\, e_i\, g_i & \text{for all } i = 1,\ldots,n-1,
\end{array}$$

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where s_i denotes the transposition $(i, i+1)$ and

$$e_i := \frac{1}{d}\sum_{k=0}^{d-1} t_i^k t_{i+1}^{d-k} \tag{2.2}$$

for all $i = 1, \ldots, n-1$. The algebra $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ was called in [ChPo1] the *affine Yokonuma–Hecke algebra* and was studied in [ChPo2], together with its cyclotomic quotients. This algebra is isomorphic to the modular framisation of the affine Hecke algebra; see definition in [JuLa2, Section 6] and Remark 1 in [ChPo1]. For $d = 1$, $\mathrm{Y}_{1,n}^{\mathrm{aff}}(q)$ is the standard affine Hecke algebra of type A.

Remark 2.1. Relations (br1), (br2), (aff1) and (aff2) are the defining relations of the *affine braid group* B_n^{aff}. Adding relations (fr1), (fr2) and (aff3) yields the definition of the *extended affine braid group* or *framed affine braid group* $\mathbb{Z}^n \rtimes B_n^{\mathrm{aff}}$, with the t_j's being interpreted as the "elementary framings" (framing 1 on the jth strand). The quotient of $\mathbb{Z}^n \rtimes B_n^{\mathrm{aff}}$ over the relations (fr3) is the *modular framed affine braid group* $(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n^{\mathrm{aff}}$ (the framing of each braid strand is regarded modulo d). Thus, the affine Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ can be obtained as the quotient of the group algebra $\mathcal{R}[(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n^{\mathrm{aff}}]$ over the quadratic relation (quad).

Note that, for all $i = 1, \ldots, n-1$, the elements e_i are idempotents, and we have $g_i e_i = e_i g_i$. Moreover, the elements g_i are invertible, with

$$g_i^{-1} = g_i - (q - q^{-1})\, e_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{2.3}$$

Now, for $i, l = 1, \ldots, n$, we set

$$e_{i,l} := \frac{1}{d}\sum_{k=0}^{d-1} t_i^k t_l^{d-k} \tag{2.4}$$

The elements $e_{i,l}$ are idempotents. We also have $e_{i,i} = 1$, $e_{i,i+1} = e_i$ and $e_{i,l} = e_{l,i}$. Moreover, it is easy to check that

$$t_j e_{i,l} = t_{s_{i,l}(j)} e_{i,l} = e_{i,l} t_{s_{i,l}(j)} = e_{i,l} t_j \qquad \text{for all } j = 1, \ldots, n, \tag{2.5}$$

where $s_{i,l}$ denotes the transposition (i, l).

We define inductively elements $X_2, \ldots, X_n$ of $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ by

$$X_{i+1} := g_i X_i g_i \quad \text{for } i = 1, \ldots, n-1. \tag{2.6}$$

We have that the elements $t_1, \ldots, t_n, X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ form a commutative family [ChPo1, Proposition 1]. Moreover, in [ChPo1, Lemma 1], it is proved that, for any $i \in \{1, \ldots, n\}$, we have

$$g_j X_i = X_i g_j \quad \text{and} \quad g_j X_i^{-1} = X_i^{-1} g_j \quad \text{for } j = 1, \ldots, n-1 \text{ such that } j \neq i-1, i. \tag{2.7}$$

Finally, using (2.1)(quad) and (2.3), it is easy to check that, for any $i \in \{1, \ldots, n\}$, we have

$$g_i X_i = X_{i+1} g_i - (q - q^{-1}) e_i X_{i+1} \quad \text{and} \quad g_i X_{i+1} = X_i g_i + (q - q^{-1}) e_i X_{i+1}\,, \tag{2.8}$$

which in turn yields

$$g_i X_i^{-1} = X_{i+1}^{-1} g_i + (q - q^{-1}) e_i X_i^{-1} \quad \text{and} \quad g_i X_{i+1}^{-1} = X_i^{-1} g_i - (q - q^{-1}) e_i X_i^{-1}\,. \tag{2.9}$$

We can easily prove by induction, on $a, b \in \mathbb{Z}_{>0}$, that the following equalities hold:

$$g_i X_i^a = X_{i+1}^a g_i - (q - q^{-1}) e_i \sum_{k=0}^{a-1} X_i^k X_{i+1}^{a-k} \quad \text{and} \quad g_i X_{i+1}^b = X_i^b g_i + (q - q^{-1}) e_i \sum_{k=0}^{b-1} X_i^k X_{i+1}^{b-k} \tag{2.10}$$

$$g_i X_i^{-a} = X_{i+1}^{-a} g_i + (q - q^{-1}) e_i \sum_{k=0}^{a-1} X_i^{-a+k} X_{i+1}^{-k} \quad \text{and} \quad g_i X_{i+1}^{-b} = X_i^{-b} g_i - (q - q^{-1}) e_i \sum_{k=0}^{b-1} X_i^{-b+k} X_{i+1}^{-k} \tag{2.11}$$

Note also that

$$g_i X_i X_{i+1} = g_i X_i g_i X_i g_i = X_{i+1} X_i g_i = X_i X_{i+1} g_i \quad \text{and} \quad g_i X_i^{-1} X_{i+1}^{-1} = X_i^{-1} X_{i+1}^{-1} g_i. \tag{2.12}$$

The above formulas yield it turn the following lemma:

Lemma 2.2. *We have the following identities satisfied in* $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$ $(i=1,\dots,n-1)$*:*

$$(2.13)\qquad g_i X_i^a X_{i+1}^b = \begin{cases} X_i^b X_{i+1}^a g_i - (q-q^{-1}) e_i \sum_{k=0}^{a-b-1} X_i^{b+k} X_{i+1}^{a-k} & \text{if } a \geq b, \\ X_i^b X_{i+1}^a g_i + (q-q^{-1}) e_i \sum_{k=0}^{b-a-1} X_i^{a+k} X_{i+1}^{b-k} & \text{if } a \leq b, \end{cases} \qquad a,b\in\mathbb{Z}\,.$$

Let $w \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters, and let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a reduced expression for w. Since the generators g_i of $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$ satisfy the same braid relations, (br1) and (br2), as the generators of $\mathfrak{S}_n$, Matsumoto's lemma implies that the element $g_w := g_{i_1} g_{i_2} \dots g_{i_r}$ is well-defined, that is, it does not depend on the choice of the reduced expression for w. We then obtain an $\mathcal{R}$-basis of $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$ as follows:

Theorem 2.3. [ChPo2, Theorem 4.15] *The set*

$$\mathcal{B}^{\mathrm{aff}}_{d,n} = \left\{ t_1^{a_1} \cdots t_n^{a_n}\, X_1^{b_1} \cdots X_n^{b_n}\, g_w \mid a_1,\dots,a_n \in \mathbb{Z}/d\mathbb{Z},\ b_1,\dots,b_n\in\mathbb{Z},\ w\in\mathfrak{S}_n \right\}$$

is an $\mathcal{R}$*-basis of* $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$*.*

For $d=1$, $\mathcal{B}^{\mathrm{aff}}_{1,n}$ is the standard Bernstein basis of the affine Hecke algebra $\mathrm{Y}^{\mathrm{aff}}_{1,n}(q)$ of type A.

3. The pro-p-Iwahori–Hecke algebra

Let Λ denote the abelian group $\mathbb{Z}^n$. For $\lambda = (\lambda_1,\dots,\lambda_n)$, $\lambda' = (\lambda'_1,\dots,\lambda'_n) \in \Lambda$, we have $\lambda\lambda' = \lambda'\lambda = (\lambda_1+\lambda'_1,\dots,\lambda_n+\lambda'_n)$. We denote by $\lambda \circ \lambda'$ the dot product of λ and λ', that is, the integer $\sum_{j=1}^n \lambda_j \lambda'_j$. We set $\varepsilon_i := (0,0,\dots,0,1,-1,0,\dots,0) \in \Lambda$, where 1 is in the i-th position and -1 is in the $(i+1)$-th position, for $i=1,\dots,n-1$.

Let W be the extended affine Weyl group $\Lambda \rtimes \mathfrak{S}_n$ of type A. For all $\sigma \in \mathfrak{S}_n$ and $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda$, we set

$$\sigma(\lambda) := \sigma\lambda\sigma^{-1} = (\lambda_{\sigma(1)},\dots,\lambda_{\sigma(n)}).$$

Now, the symmetric group $\mathfrak{S}_n$ is generated by the set $S = \{s_1,\dots,s_{n-1}\}$, where s_i denotes the transposition $(i,i+1)$. We set

$$\gamma := s_{n-1}s_{n-2}\dots s_2 s_1 \in \mathfrak{S}_n \quad\text{and}\quad h := (0,0,\dots,0,1)\,\gamma \in W.$$

Note that we have $hs_ih^{-1} = s_{i-1}$ for all $i=2,\dots,n-1$. Set $s_0 := hs_1h^{-1} \in W$. Then the set $S^{\mathrm{aff}} = S \cup \{s_0\}$ is a generating set of the affine Weyl group W^{aff} of type A and we have $W = \langle h \rangle \ltimes W^{\mathrm{aff}}$. Moreover, we can extend the length function ℓ of W^{aff} to W by setting $\ell(h^k w^{\mathrm{aff}}) = \ell(w^{\mathrm{aff}})$ for all $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$, $k\in\mathbb{Z}$.

Let $\mathcal{T}$ be a (finite) abelian group such that $\mathfrak{S}_n$ acts on $\mathcal{T}$. Like above, for all $\sigma\in\mathfrak{S}_n$ and $t\in\mathcal{T}$, we set $\sigma(t) := \sigma t\sigma^{-1}$.

We consider the semi-direct product $\widetilde{W} := (\mathcal{T}\times\Lambda)\rtimes\mathfrak{S}_n$. Every element of $\widetilde{W}$ can be written in the form $t\,\lambda\,\sigma$, with $t\in\mathcal{T}$, $\lambda\in\Lambda$, $\sigma\in\mathfrak{S}_n$. Note that t and λ commute with each other. We set $\widetilde{\Lambda} := \mathcal{T}\times\Lambda$ and $\widetilde{\mathfrak{S}}_n := \mathcal{T}\rtimes\mathfrak{S}_n$. We have $\widetilde{W} = \widetilde{\Lambda}\widetilde{\mathfrak{S}}_n$, with $\widetilde{\Lambda}\cap\widetilde{\mathfrak{S}}_n = \mathcal{T}$. Like above, for all $\sigma\in\mathfrak{S}_n$ and $\nu\in\widetilde{\Lambda}$, we set $\sigma(\nu) := \sigma\nu\sigma^{-1}$. We also set $\widetilde{S} := \{ts \mid s\in S,\, t\in\mathcal{T}\}$ and $\widetilde{S}^{\mathrm{aff}} := \{ts \mid s\in S^{\mathrm{aff}},\, t\in\mathcal{T}\}$. Finally, we can extend the length function ℓ of W to $\widetilde{W}$ by setting $\ell(tw) = \ell(w)$ for all $w\in W$, $t\in\mathcal{T}$.

Theorem 3.1. [Vi2, Theorem 2.4] *Let R be a ring and let* $(q_s,c_s)_{s\in\widetilde{S}^{\mathrm{aff}}} \in R\times R[\mathcal{T}]$ *be such that*

(a) $q_s = q_{ts}$ *and* $c_{ts} = tc_s$ *for all* $t\in\mathcal{T}$*.*

(b) $q_s = q_{s'}$ *and* $c_{s'} = wc_sw^{-1}$ *if* $s' = wsw^{-1}$ *for some* $w\in\widetilde{W}$*.*

Then the free R-module $\mathcal{H}_R(q_s,c_s)$ *of basis* $(T_w)_{w\in\widetilde{W}}$ *has a unique R-algebra structure satisfying:*

- *The braid relations:* $T_wT_{w'} = T_{ww'}$ *if* $w,w'\in\widetilde{W}$, $\ell(w)+\ell(w') = \ell(ww')$*.*
- *The quadratic relations:* $T_s^2 = q_ss^2 + c_sT_s$ *for* $s\in\widetilde{S}^{\mathrm{aff}}$*.*

The algebra $\mathcal{H}_R(q_s, c_s)$ is called the *pro-p-Iwahori–Hecke algebra of* $\widetilde{W}$ over R. If q_s is invertible in R for all $s \in \widetilde{S}^{\mathrm{aff}}$, then T_s is invertible in $\mathcal{H}_R(q_s, c_s)$, with

$$T_s^{-1} = q_s^{-1} s^{-2} \left(T_s - c_s \right). \tag{3.1}$$

We deduce that every element T_w, for $w \in \widetilde{W}$, is invertible in $\mathcal{H}_R(q_s, c_s)$. If, further, $q_s^{1/2} \in R$ for all $s \in \widetilde{S}^{\mathrm{aff}}$, then, by replacing the generators T_s by $\bar{T}_s := q_s^{-1/2} T_s$, the quadratic relations become

$$\bar{T}_s^2 = s^2 + q_s^{-1/2} c_s \bar{T}_s \quad \text{for all} \quad s \in \widetilde{S}^{\mathrm{aff}}.$$

Thus, if $(q_s^{1/2})_{s \in \widetilde{S}^{\mathrm{aff}}}$ are chosen so that they also satisfy conditions (a) and (b) of Theorem 3.1, we obtain an isomorphism between $\mathcal{H}_R(q_s, c_s)$ and $\mathcal{H}_R(1, q_s^{-1/2} c_s)$. Therefore, without loss of generality, we may assume that $q_s = 1$ for all $s \in \widetilde{S}^{\mathrm{aff}}$.

We now have the following Bernstein presentation of $\mathcal{H}_R(1, c_s)$.

Theorem 3.2. [Vi2, Theorem 2.10] *The R-algebra $\mathcal{H}_R(1, c_s)$ is isomorphic to the free R-module of basis $(E(w))_{w \in \widetilde{W}}$ endowed with the unique R-algebra structure satisfying:*

- *Braid relations: $E(w)E(w') = E(ww')$ for $w, w' \in \widetilde{\mathfrak{S}}_n$, $\ell(w) + \ell(w') = \ell(ww')$.*
- *Quadratic relations: $E(s)^2 = s^2 + c_s E(s)$ for $s \in \widetilde{S}$.*
- *Product relations: $E(\nu)E(w) = E(\nu w)$ for $\nu \in \widetilde{\Lambda}$, $w \in \widetilde{W}$.*
- *Bernstein relations: For $s = ts_i \in \widetilde{S}$ ($t \in \mathcal{T}$, $i = 1, \ldots, n-1$) and $\nu = \tau\lambda \in \widetilde{\Lambda}$ ($\tau \in \mathcal{T}$, $\lambda \in \Lambda$),*

$$E(s(\nu))E(s) - E(s)E(\nu) = \begin{cases} 0 & \text{if } s_i(\lambda) = \lambda, \\ \epsilon_i(\lambda)\, c_s \sum_{k=0}^{|\varepsilon_i \circ \lambda| - 1} E(\tau \mu_i(k, \lambda)) & \text{if } s_i(\lambda) \neq \lambda, \end{cases}$$

where $\epsilon_i(\lambda) = \mathrm{sign}(\varepsilon_i \circ \lambda) \in \{1, -1\}$ and

$$\mu_i(k, \lambda) = \begin{cases} \lambda\, \varepsilon_i^k & \text{if } \epsilon_i(\lambda) = -1 \\ s_i(\lambda)\, \varepsilon_i^k & \text{if } \epsilon_i(\lambda) = 1\,. \end{cases}$$

Remark 3.3. Note that, in the above presentation, we take $E(s) = T_s$ for all $s \in \widetilde{S}$ (in particular, $E(t) = t$ for all $t \in \mathcal{T}$).

4. Main result

Let q be an indeterminate or a non-zero complex number. Our aim in this section will be to show that the affine Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}^{\mathrm{aff}}(q)$ is isomorphic to the pro-p-Iwahori–Hecke algebra $\mathcal{H}_R(q_s, c_s)$ of $\widetilde{W}$ when we take

- $\mathcal{T} = (\mathbb{Z}/d\mathbb{Z})^n = \langle\, t_1, \ldots, t_n \mid t_j^d = 1,\ t_i t_j = t_j t_i, \text{ for all } i, j = 1, \ldots, n \,\rangle$;
- $R = \mathcal{R} = \mathbb{C}[q, q^{-1}]$;
- $q_{ts_i} = 1$ for all $i = 0, 1, \ldots, n-1$, $t \in (\mathbb{Z}/d\mathbb{Z})^n$;
- $c_{ts_i} = (q - q^{-1})\, t\, e_i$ for all $i = 0, 1, \ldots, n-1$, $t \in (\mathbb{Z}/d\mathbb{Z})^n$, where $e_0 := e_{1,n}$.

We then have $\widetilde{W} = (\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z})^n \rtimes \mathfrak{S}_n = \{t\,\lambda\,\sigma \,|\, t \in (\mathbb{Z}/d\mathbb{Z})^n,\ \lambda \in \Lambda,\ \sigma \in \mathfrak{S}_n\}$. The action of $\mathfrak{S}_n$ on $(\mathbb{Z}/d\mathbb{Z})^n$ is given by

$$\sigma(t_j) = \sigma t_j \sigma^{-1} = t_{\sigma(j)} \qquad \text{for all } \sigma \in \mathfrak{S}_n,\ j = 1, \ldots, n.$$

The action of $\mathfrak{S}_n$ extends linearly to the group algebra $\mathcal{R}[(\mathbb{Z}/d\mathbb{Z})^n]$.

First we check that the assumptions (a) and (b) of Theorem 3.1 are satisfied in this case. Let $s \in \widetilde{S}^{\mathrm{aff}} = \{ts_i \,|\, i = 0, 1, \ldots, n-1, t \in (\mathbb{Z}/d\mathbb{Z})^n\}$. By definition, we have $q_s = q_{ts} = 1$ and $c_{ts} = tc_s$ for all $t \in (\mathbb{Z}/d\mathbb{Z})^n$.

Now, set

$$\lambda_0 := (-1, 0, \ldots, 0, 1) \in \Lambda \quad \text{and} \quad \lambda_i := (0, 0, \ldots, 0, 0) \in \Lambda \quad \text{for all } i = 1, \ldots, n-1.$$

Moreover, let σ_0 denote the transposition $(1, n) = s_{1,n}$ and σ_i denote the transposition $(i, i+1) = s_i$ for all $i = 1, \ldots, n-1$. We then have

$$s_i = \lambda_i \sigma_i \quad \text{for all } i = 0, 1, \ldots, n-1.$$

So

$$\widetilde{S}^{\mathrm{aff}} = \{t\lambda_i\sigma_i \,|\, i = 0, 1, \ldots, n-1, t \in (\mathbb{Z}/d\mathbb{Z})^n\}.$$

Let $s, s' \in \widetilde{S}^{\mathrm{aff}}$ be conjugate in $\widetilde{W}$. By definition, we have $q_s = q_{s'} = 1$. Now, let us write $s = t\lambda_i\sigma_i$ and $s' = t'\lambda_{i'}\sigma_{i'}$ for $i, i' \in \{0, 1, \ldots, n-1\}$ and $t, t' \in (\mathbb{Z}/d\mathbb{Z})^n$. Moreover, let $w \in \widetilde{W}$ be such that $s' = wsw^{-1}$, and write $w = \tau\lambda\sigma$ with $\tau \in (\mathbb{Z}/d\mathbb{Z})^n$, $\lambda \in \Lambda$ and $\sigma \in \mathfrak{S}_n$. Then

$$t'\lambda_{i'}\sigma_{i'} = (\tau\lambda\sigma)(t\lambda_i\sigma_i)(\sigma^{-1}\lambda^{-1}\tau^{-1}) = [\tau\sigma(t)(\sigma\sigma_i\sigma^{-1})(\tau^{-1})][\lambda\sigma(\lambda_i)(\sigma\sigma_i\sigma^{-1})(\lambda^{-1})][\sigma\sigma_i\sigma^{-1}].$$

We deduce that

$$\sigma_{i'} = \sigma\sigma_i\sigma^{-1} \quad \text{and} \quad t' = \tau\sigma(t)(\sigma\sigma_i\sigma^{-1})(\tau^{-1}) = \tau\sigma(t)\sigma_{i'}(\tau^{-1}).$$

By (2.5), we have

$$t'e_{i'} = \tau\sigma(t)\sigma_{i'}(\tau^{-1})e_{i'} = \tau\sigma(t)\tau^{-1}e_{i'} = \sigma(t)e_{i'} = \sigma(t)ww^{-1}e_{i'} = wtw^{-1}e_{i'}. \tag{4.1}$$

Furthermore, we have

$$\sigma_{i'} = \sigma\sigma_i\sigma^{-1} = s_{\sigma(i),\sigma(i+1)} = (\sigma(i), \sigma(i+1)),$$

which implies that

$$we_iw^{-1} = \sigma(e_i)ww^{-1} = \sigma(e_i) = e_{\sigma(i),\sigma(i+1)} = e_{i'}. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$c_{s'} = (q - q^{-1})t'e_{i'} = (q - q^{-1})wtw^{-1}we_iw^{-1} = w\left((q - q^{-1})te_i\right)w^{-1} = wc_sw^{-1}.$$

Therefore, we can define the pro-p-Iwahori–Hecke algebra $\mathcal{H}_{\mathcal{R}}(1, c_s)$ of $\widetilde{W}$, which is the free $\mathcal{R}$-module of basis $(E(w))_{w\in\widetilde{W}}$ endowed with the unique $\mathcal{R}$-algebra structure satisfying the relations described explicitly in the previous section.

Theorem 4.1. *The $\mathcal{R}$-linear map $\varphi : \mathrm{Y}^{\mathrm{aff}}_{d,n}(q) \to \mathcal{H}_{\mathcal{R}}(1, c_s)$ defined by*

$$\varphi(t_1^{a_1}\cdots t_n^{a_n}\, X_1^{b_1}\cdots X_n^{b_n}\, g_w) = E(t_1^{a_1}\cdots t_n^{a_n}\,(b_1, \ldots, b_n)\, w)\ , \tag{4.3}$$

for all $a_1, \ldots, a_n \in \mathbb{Z}/d\mathbb{Z}$, $b_1, \ldots, b_n \in \mathbb{Z}$, and $w \in \mathfrak{S}_n$, is an $\mathcal{R}$-algebra isomorphism.

Proof. We will show first that we can define an $\mathcal{R}$-algebra homomorphism $\varphi : \mathrm{Y}^{\mathrm{aff}}_{d,n}(q) \to \mathcal{H}_{\mathcal{R}}(1, c_s)$ given by

$$\begin{array}{rcll} \varphi(t_j) & = & E(t_j) & \text{for all } j = 1, \ldots, n, \\ \varphi(g_i) & = & E(s_i) & \text{for all } i = 1, \ldots, n-1, \\ \varphi(X_1) & = & E((1, 0, \ldots, 0)), & \\ \varphi(X_1^{-1}) & = & E((-1, 0, \ldots, 0)). & \end{array}$$

For this, it is enough to check that the defining relations (2.1) of $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$ are satisfied by the images of its generators via φ, that is, the elements

$$\varphi(t_1), \ldots, \varphi(t_n), \varphi(g_1), \ldots, \varphi(g_{n-1}), \varphi(X_1), \varphi(X_1^{-1}).$$

First of all, note that $\varphi(X_1^{-1}) = E((-1, 0, \ldots, 0)) = E((1, 0, \ldots, 0))^{-1} = \varphi(X_1)^{-1}$, due to the product relations. The product relations also imply (aff3). Moreover, due to the braid relations, we have immediately that (br1), (br2), (fr1) and (fr3) are satisfied. We also obtain

$$\varphi(g_i)\varphi(t_j) = E(s_i)E(t_j) = E(s_it_j) = E(t_{s_i(j)}s_i) = E(t_{s_i(j)})E(s_i) = \varphi(t_{s_i(j)})\varphi(g_i)$$

for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$, so (fr2) holds.

Now, for $i = 1, \ldots, n-1$, we have

$$\varphi(g_i)^2 = E(s_i)^2 = s_i^2 + c_{s_i}E(s_i) = 1 + (q - q^{-1})e_iE(s_i) = 1 + (q - q^{-1})e_i\varphi(g_i),$$

so (quad) is satisfied by $\varphi(g_i)$. We will use the Bernstein relations for the remaining defining relations, (aff1) and (aff2).

First, note that $(1, 0, \ldots, 0)$ is fixed by the action of s_i for all $i = 2, \ldots, n-1$. We thus obtain

$$\varphi(X_1)\varphi(g_i) = E((1, 0, \ldots, 0))E(s_i) = E(s_i(1, 0, \ldots, 0)s_i^{-1})E(s_i) = E(s_i)E((1, 0, \ldots, 0)) = \varphi(g_i)\varphi(X_1).$$

This yields (aff2).

Now, $(1,0,\dots,0)$ is not fixed by the action of s_1. By the Bernstein relations, we have

$$E((0,1,\dots,0))E(s_1) - E(s_1)E((1,0,\dots,0)) = c_{s_1}E((0,1,\dots,0)),$$

and thus,

$$E(s_1)E((1,0,\dots,0)) = E((0,1,\dots,0))E(s_1) - c_{s_1}E((0,1,\dots,0)) = E((0,1,\dots,0))(E(s_1) - c_{s_1}).$$

By (3.1) and Remark 3.3, we have that $E(s_1) - c_{s_1} = E(s_1)^{-1}$, and so

$$E(s_1)E((1,0,\dots,0)) = E((0,1,\dots,0))E(s_1)^{-1}. \tag{4.4}$$

We obtain

$$\begin{aligned}\varphi(X_1)\varphi(g_1)\varphi(X_1)\varphi(g_1) &= E((1,0,\dots,0))E(s_1)E((1,0,\dots,0))E(s_1)\\ &= E((1,0,\dots,0))E((0,1,\dots,0))E(s_1)^{-1}E(s_1)\\ &= E((1,0,\dots,0))E((0,1,\dots,0)).\\ &= E((1,1,\dots,0)).\end{aligned}$$

and

$$\begin{aligned}\varphi(g_1)\varphi(X_1)\varphi(g_1)\varphi(X_1) &= E(s_1)E((1,0,\dots,0))E(s_1)E((1,0,\dots,0))\\ &= E((0,1,\dots,0))E(s_1)^{-1}E(s_1)E((1,0,\dots,0))\\ &= E((0,1,\dots,0))E((1,0,\dots,0)).\\ &= E((1,1,\dots,0)).\end{aligned}$$

Thus, (aff1) also holds, and φ is an $\mathcal{R}$-algebra homomorphism.

We will now prove that φ is indeed the $\mathcal{R}$-linear map given by (4.3). First, for all $i = 1,\dots,n-1$, we have

$$\begin{aligned}\varphi(X_{i+1}) &= \varphi(g_i g_{i-1}\dots g_1 X_1 g_1 \dots g_{i-1} g_i)\\ &= \varphi(g_i)\varphi(g_{i-1})\dots\varphi(g_1)\varphi(X_1)\varphi(g_1)\dots\varphi(g_{i-1})\varphi(g_i)\\ &= E(s_i)E(s_{i-1})\dots E(s_1)E((1,0,\dots,0))E(s_1)\dots E(s_{i-1})E(s_i).\end{aligned}$$

Using repeatedly the Bernstein relations (similarly to (4.4)), we obtain that

$$\varphi(X_{i+1}) = E((0,0,\dots,0,1,0,\dots,0)),$$

where 1 is in position $i+1$. Now, by the product and braid relations, we obviously get

$$\varphi(t_1^{a_1}\cdots t_n^{a_n}\, X_1^{b_1}\cdots X_n^{b_n}\, g_w) = E(t_1^{a_1}\cdots t_n^{a_n}\,(b_1,\dots,b_n)\, w),$$

for all $a_1,\dots,a_n \in \mathbb{Z}/d\mathbb{Z}$, $b_1,\dots,b_n \in \mathbb{Z}$, and $w \in \mathfrak{S}_n$.

Finally, by Theorem 2.3 and Theorem 3.2, the map φ is bijective, so φ is an $\mathcal{R}$-algebra isomorphism, as required. □

Now, if q^2 is a power of a prime number p and $d = q^2 - 1$, then the pro-p-Iwahori–Hecke algebra $\mathcal{H}_{\mathcal{R}}(1, c_s)$ defined above is the convolution algebra of complex valued and compactly supported functions on the group $\mathrm{GL}_n(F)$, with F a suitable p-adic field, that are bi-invariant under the pro-p-radical of an Iwahori subgroup [Vi1, Theorem 1]. Therefore, the following result, also stated in [ChPo2, Remark 2.2], is an immediate consequence of Theorem 4.1.

Corollary 4.2. *If $q^2 = p^k$, where p is a prime number and $k \in \mathbb{Z}_{>0}$, and $d = q^2 - 1$, then the affine Yokonuma–Hecke algebra $\mathrm{Y}^{\mathrm{aff}}_{d,n}(q)$ is isomorphic to the convolution algebra of complex valued and compactly supported functions on the group $\mathrm{GL}_n(F)$, where F is a local non-archimedean field of residual field $\mathbb{F}_{q^2}$, that are bi-invariant under the pro-p-radical of an Iwahori subgroup.*

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