ℓ-MODULAR REPRESENTATIONS OF p-ADIC GROUPS ($\ell \neq p$)

by

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Introduction

These are notes from a series of lectures that I gave in April 2013 at the Institute of Mathematics of Singapore. I thank the organizers for the opportunity to give these lectures. They were concerned with the representation theory of reductive p-adic groups with coefficients in a field of characteristic $\ell \neq 0, p$.

The representation theory of reductive p-adic groups with coefficients in the field of complex numbers has been developed since the 1960's (see Cartier's introduction [11]). It has inherited certain techniques coming from harmonic analysis on reductive groups over Archimedean fields, making a large use of the fact that the representations have complex coefficients (Harish-Chandra [16], Langlands' classification [25]). Then in the 1980's, Bernstein and Zelevinski developed a fully algebraic approach [2, 3, 4].

The arithmetic of modular forms has required to develop the representation theory of reductive p-adic groups with coefficients in fields, or even rings, other than complex numbers. Provided that p is invertible in the coefficient ring, a large part of Bernstein-Zelevinski's algebraic approach can be reproduced (Vignéras [27, 28]). In these lectures, I will assume that the coefficient ring is an algebraically closed field with characteristic different from p. For bibliographic references, see Vignéras [27] and Blondel [5].

In Lecture 1, I define parabolic induction and restriction, and the notions of cuspidal representation and cuspidal support. An important aspect of the theory of ℓ-modular representation is that there is a difference between cuspidal and supercuspidal representations (Example 1.11). This leads to the notion of supercuspidal support and to the problem of classifying all irreducible representations with a given supercuspidal support.

In Lecture 2, I discuss the case of the group GL_n and its inner forms. I explain how, thanks to the theory of types developed by C. Bushnell and P. Kutzko, one can prove the uniqueness of supercuspidal support for irreducible representations of these groups (Theorem 2.1).

In Lecture 3, I present the classification of all irreducible representations of $\mathrm{GL}_n(\mathrm{F})$ in terms of multisegments, generalizing Zelevinski's classification of complex irreducible representations. First, type theory allows one to reduce to the classification of all unipotent irreducible representations. Then one defines a map:

$$\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m}) \tag{0.1}$$

that associates a unipotent irreducible representation to any multisegment. The definition of this map and the proof that it is injective relies on the theory of generic representations for the group $\mathrm{GL}_n(\mathrm{F})$ (see also Remark 3.11 for inner forms). Proof of surjectivity requires a counting argument that relies on results of [1, 12] on the classification of simple modules over an affine Hecke algebra of type A at a root of unity.

In Lecture 4, I introduce the operation of reducing mod ℓ an irreducible ℓ-adic representation of G having a stable lattice. Then I present the properties of the bijection (0.1) and of the local Langlands correspondence with respect to reduction mod ℓ.

Lecture 1.

1.1. Notation and preliminaries

In all these lectures, we fix a locally compact non-Archimedean field F of residue characteristic denoted p ; we write $\mathcal{O}$ for its ring of integers, $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}$ and q for the cardinality of its residue field. We also fix an algebraically closed field R of characteristic not dividing q.

Let $\mathbf{G}$ be a connected reductive group defined over F, and let $\mathrm{G} = \mathbf{G}(\mathrm{F})$ be the group of its F-points. When endowed with the topology coming from that of F, the group G is locally compact, and its neutral element 1 has a basis of neighborhoods made of compact open pro-p-subgroups (that is, all of whose open subgroups have index of the form p^r, $r \geqslant 0$).

Example 1.1. — If $\mathbf{G} = \mathrm{GL}_n$, then G is the group $\mathrm{GL}_n(\mathrm{F})$. The identity matrix has a basis of neighborhoods made of the compact open pro-p-subgroups $\mathrm{K}_i = 1 + \mathrm{M}_n(\mathfrak{p}^i)$ for $i \geqslant 1$.

Definition 1.2. — A *smooth* R*-representation of* G is a pair (π, V) made of a vector space V over R together with a group homomorphism $\pi : \mathrm{G} \mapsto \mathrm{GL}(\mathrm{V})$ such that, for all $v \in \mathrm{V}$, there is a compact open subgroup of G fixing v.

In these lectures, all representations will be smooth R-representations. Therefore we will often write *representation* for *smooth* R*-representation*.

Given two smooth R-representations (π, V) and (σ, W) of G, a homomorphism from (π, V) to (σ, W) is an R-linear map $f : \mathrm{V} \to \mathrm{W}$ such that $f \circ \pi(g) = \sigma(g) \circ f$ for all $g \in \mathrm{G}$. The space of all such maps will be denoted $\mathrm{Hom}_{\mathrm{G}}(\pi, \sigma)$.

This defines the abelian category, denoted $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$, of smooth R-representations of G.

Definition 1.3. — A smooth R-representation (π, V) of G is said to be *admissible* if, for any open subgroup H of G, the space V^{H} of H-fixed vectors of V is finite-dimensional.

A smooth R-character (or character for short) of G is a group homomorphism from G to $\mathrm{R}^\times$ with open kernel.

Given a representation π and a character χ of G, we write $\pi\chi$ for the twisted representation $g \mapsto \pi(g)\chi(g)$.

A first very important fact is that one can define R-valued Haar measures on the group G (see [27, I.2]). Such a measure is a nonzero R-linear form on:

$$\mathcal{C}_c^\infty(\mathrm{G}, \mathrm{R}) = \{\text{compactly supported and locally constant R-valued functions on G}\}$$

which is invariant under right translations (it is then automatically invariant under left translations). Such a measure is unique up to a non zero scalar. To define a Haar measure, let us fix a compact open pro-p-subgroup $\mathrm{H} \subseteq \mathrm{G}$ and define the measure of any compact open subgroup K of G by:

$$\mathrm{meas}(\mathrm{K}) = \frac{(\mathrm{K} : \mathrm{K} \cap \mathrm{H})}{(\mathrm{H} : \mathrm{K} \cap \mathrm{H})} \in \mathrm{R},$$

which is well defined since the denominator is a p-power and p is invertible in R.

We have here our first important difference between the complex theory (when R is the field $\mathbf{C}$ of complex numbers) and the modular theory (when R has characteristic $\ell > 0$). Unlike the complex case, meas(K) may be 0 in the modular case. Smooth R-representations of K need not be semi-simple, and the functor $\mathrm{V} \mapsto \mathrm{V}^{\mathrm{K}}$ of K-fixed vectors need not be exact.

1.2. Parabolic functors

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, together with a Levi decomposition $\mathbf{P} = \mathbf{MN}$ defined over F, where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$. Write $\mathrm{P} = \mathbf{P}(\mathrm{F})$, $\mathrm{M} = \mathbf{M}(\mathrm{F})$ and $\mathrm{N} = \mathbf{N}(\mathrm{F})$.

Attached to the parabolic subgroup P there is a complex character δ_{P} of M defined by:

$$\delta_{\mathrm{P}}(m) = (m\mathrm{K}m^{-1} : m\mathrm{K}m^{-1} \cap \mathrm{K})/(\mathrm{K} : m\mathrm{K}m^{-1} \cap \mathrm{K})$$

for all $m \in \mathrm{M}$, where K is an arbitrary compact open pro-p-subgroup of N (the value $\delta_{\mathrm{P}}(m)$ does not depend on the choice of K). These values are integer powers of q. Thus for all $m \in \mathrm{M}$, there is a $v(m) \in \mathbf{Z}$ such that $\delta_{\mathrm{P}}(m) = q^{v(m)}$. Let us make a choice of a square root of q in R, denoted $\sqrt{q}$, and write:

$$\sqrt{\delta_{\mathrm{P}}} : m \mapsto (\sqrt{q})^{v(m)} \in \mathrm{R}^{\times}.$$

Let (σ, W) be a smooth representation of M. Write $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}(\mathrm{W})$ for the space of locally constant functions $f : \mathrm{G} \to \mathrm{W}$ such that:

$$f(mng) = \sqrt{\delta_{\mathrm{P}}}(m)\sigma(m)f(g), \quad m \in \mathrm{M}, n \in \mathrm{N}, g \in \mathrm{G}.$$

The representation of G on this space by right translation is smooth, denoted $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}(\sigma)$ and called the *parabolic induction of* (σ, W) to G along P. The functor:

$$\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}} : \mathscr{R}_{\mathrm{R}}(\mathrm{M}) \to \mathscr{R}_{\mathrm{R}}(\mathrm{G})$$

has a left adjoint $\boldsymbol{r}_{\mathrm{M,P}}^{\mathrm{G}}$, the *parabolic restriction* functor from G to M along P. Given (π, V) a smooth representation of G, write V(N) for the subspace of V spanned by $\pi(n)v - v$ for all $n \in \mathrm{N}$ and $v \in \mathrm{V}$. Then $\boldsymbol{r}_{\mathrm{M,P}}^{\mathrm{G}}(\pi)$ is the natural representation of M on the quotient V/V(N) twisted by the inverse of the character $\sqrt{\delta_{\mathrm{P}}}$.

Remark 1.4. — (1) The functors $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$, $\boldsymbol{r}_{\mathrm{M,P}}^{\mathrm{G}}$ are exact, $\boldsymbol{r}_{\mathrm{M,P}}^{\mathrm{G}}$ preserves the property of being of finite type, and $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$ preserves admissibility (see [27, II.2.1]).

(2) More difficult: $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$ and $\boldsymbol{r}_{\mathrm{M,P}}^{\mathrm{G}}$ preserve the property of having finite length ([27, II.5.13]).

We have the following theorem (see [27], II.2.18 for more details).

***Theorem 1.5* (Geometric Lemma).** — *Given parabolic subgroups* $\mathrm{P} = \mathrm{MN}$ *and* $\mathrm{Q} = \mathrm{LU}$ *of* G*, there is a formula describing the functor* $\boldsymbol{r}_{\mathrm{L,Q}}^{\mathrm{G}} \circ \boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$.

If R is the field of complex numbers, $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$ has a right adjoint, which is $\boldsymbol{r}_{\mathrm{M,P^-}}^{\mathrm{G}}$ with P^- the parabolic subgroup of G opposite to P with respect to M (this is known as the second adjointness property; see [7]). In the modular case, this is not known in general, but partial results can be found in [13]. However, there is a version for admissible representations (see [27, II.3.8]).

1.3. Cuspidal representations

Definition 1.6. — A representation (π, V) of G is *cuspidal* if the following equivalent conditions are satisfied:

(1) The space $\boldsymbol{r}^{\mathrm{G}}_{\mathrm{M},\mathrm{P}}(\mathrm{V})$ is zero for all proper parabolic subgroups $\mathrm{P} = \mathrm{MN} \subsetneq \mathrm{G}$.

(2) The space $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}, \boldsymbol{i}^{\mathrm{G}}_{\mathrm{M},\mathrm{P}}(\mathrm{W}))$ is zero for all smooth R-representations (σ, W) of M and all proper parabolic subgroups $\mathrm{P} = \mathrm{MN} \subsetneq \mathrm{G}$.

Theorem 1.7 **([27, 22])**. — *Given* (π, V) *an irreducible representation of* G*, there are a parabolic subgroup* $\mathrm{P} = \mathrm{MN} \subseteq \mathrm{G}$ *and an irreducible cuspidal representation* (σ, W) *of* M *such that* π *embeds in* $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{M},\mathrm{P}}(\sigma)$*. Moreover, the cuspidal pair* (M, σ) *is unique up to* G*-conjugacy.*

Definition 1.8. — The G-conjugacy class of (M, σ) is called the *cuspidal support* of (π, V), denoted $\mathrm{cusp}(\pi, \mathrm{V})$.

The cuspidal support depends on the choice of $\sqrt{q} \in \mathrm{R}^{\times}$ that we have made.

Problem 1: Classify all irreducible representations of G having given cuspidal support.

There ia another characterization of cuspidality. Write Z for the centre of G.

Proposition 1.9. — *An irreducible representation* (π, V) *is cuspidal if and only if, for all* $v \in \mathrm{V}$ *and all smooth linear form* $\xi : \mathrm{V} \to \mathrm{R}$*, the function* $g \mapsto \xi(\pi(g)v)$ *has support whose image in* G/Z *is compact.*

Corollary 1.10 **([27])**. — *All irreducible representation of* G *are admissible and have a central character.*

When $\mathrm{R} = \mathbf{C}$, Proposition 1.9 is crucial. It is one of the key properties used for the Bernstein decomposition of the category $\mathscr{R}_{\mathbf{C}}(\mathrm{G})$ into blocks with respect to the notion of (inertial) cuspidal support. This is related to the fact that an irreducible cuspidal representation of G with central character ω is projective in the full subcategory of $\mathscr{R}_{\mathbf{C}}(\mathrm{G})$ made of all smooth R-representations having central character ω.

As observed by Vignéras, this is no longer true in the modular case, since irreducible cuspidal representations of G may occur as subquotients of proper parabolically induced representations.

Example 1.11. — Assume $\mathrm{G} = \mathrm{GL}_2(\mathbf{Q}_5)$ and $\ell = 3$ (where $\mathbf{Q}_p$ is the field of p-adic numbers). Let B be the subgroup of upper triangular matrices. The representation of G on the space V of R-valued locally constant functions on $\mathrm{B}\backslash\mathrm{G}$ is indecomposable and has length 3. Its unique irreducible subrepresentation is the trivial character of G, and its unique irreducible quotient is $g \mapsto |\det g|$, where $x \mapsto |x|$ denotes the absolute value of F giving value q^{-1} to any uniformizer. The remaining (infinite-dimensional) subquotient is cuspidal.

Definition 1.12. — An irreducible representation of G is *supercuspidal* if for all proper $\mathrm{P} = \mathrm{MN}$ and all irreducible representation (σ, W) of M, it does not occur as a subquotient of $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{M},\mathrm{P}}(\sigma)$.

All supercuspidal representations of G are cuspidal, but the converse need not be true (see Example 1.11).

Remark 1.13. — When $\mathrm{R} = \mathbf{C}$, any cuspidal representation is supercuspidal.

Remark 1.14. — (1) Assume (π, V) is a supercuspidal irreducible representation of G having a projective cover P_π of finite type. Then P_π is cuspidal (by Frobenius reciprocity). This implies that π does not occur as a subquotient of $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}(\sigma)$ for any smooth (σ, W).

(2) It is not known in general whether or not a supercuspidal irreducible representation of G has a projective cover of finite type. This is known for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$ ([14]).

Proposition 1.15. — *For all irreducible representation (π, V) of G, there are a parabolic subgroup $\mathrm{P} = \mathrm{MN}$ and a supercuspidal irreducible representation (σ, W) of M such that π occurs as a subquotient of $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}(\sigma)$.*

Let us denote by $\mathrm{scusp}(\pi, \mathrm{V})$ the set of all possible such pairs (M, σ), called the *supercuspidal support* of (π, V).

Problem 2: Given an irreducible representation (π, V) of G, is $\mathrm{scusp}(\pi, \mathrm{V})$ made of a single G-conjugacy class?

The answer is known only for the groups $\mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$ and their inner forms (see Lecture 2). See also [19] for the unitary group $\mathrm{U}(2, 1)$ with respect to an unramified quadratic extension.

Lecture 2.

From now on, we will assume that $\mathbf{G}$ is GL_n, $n \geqslant 1$ or possibly one of its inner form, so that G is of the form $\mathrm{GL}_m(\mathrm{D})$ with D a central division F-algebra of degree d^2, with $n = md$.

Let $\alpha = (m_1, \ldots, m_r)$ be a family of positive integers of sum m. Denote by M_α the subgroup of $\mathrm{GL}_m(\mathrm{D})$ of invertible matrices which are diagonal by blocks of size $m_1, \ldots, m_r$ respectively (it is isomorphic to $\mathrm{GL}_{m_1}(\mathrm{D}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{D})$) and by P_α the subgroup of $\mathrm{GL}_m(\mathrm{D})$ generated by M_α and the upper triangular matrices.

Write $\boldsymbol{i}_\alpha$ for the functor of parabolic induction associated with M_α and P_α, and $\boldsymbol{r}_\alpha$ for its left adjoint. If $\pi_1, \ldots, \pi_r$ are smooth representations of $\mathrm{GL}_{m_1}(\mathrm{D}), \ldots, \mathrm{GL}_{m_r}(\mathrm{D})$ respectively, write:

$$\pi_1 \times \pi_2 \times \cdots \times \pi_r = \boldsymbol{i}_\alpha(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r).$$

If (M, σ) is a cuspidal pair, then up to conjugacy we have $\mathrm{M} = \mathrm{M}_\alpha$ for some α as above and σ has the form $\sigma_1 \otimes \cdots \otimes \sigma_r$ where σ_i is a cuspidal irreducible representation of $\mathrm{GL}_{m_i}(\mathrm{D})$. Therefore the $\mathrm{GL}_m(\mathrm{D})$-conjugacy class of (M, σ) will be identified with the formal sum $\sigma_1 + \cdots + \sigma_r$.

2.1. Supercuspidal support

Theorem 2.1 **([28, 22]).** — *Assume G is $\mathrm{GL}_n(\mathrm{F})$ or one of its inner forms. For all irreducible representations (π, V) of G, the set $\mathrm{scusp}(\pi, \mathrm{V})$ is a single G-conjugacy class.*

We need to introduce Bushnell-Kutzko's theory of types [9]. This is a monumental machinery, initially developed by Bushnell and Kutzko in order to prove that any complex irreducible cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$ is compactly induced from an irreducible representation of a compact mod centre, open subgroup of $\mathrm{GL}_n(\mathrm{F})$. More precisely:

Theorem 2.2 **([9, 27, 23]).** — *Assume G is $\mathrm{GL}_n(\mathrm{F})$ or one of its inner forms. Then there is a family of pairs (J, λ) made of a compact open subgroup $\mathrm{J} \subseteq \mathrm{G}$ and an irreducible representation λ of J with the following properties:*

(1) *For any irreducible cuspidal representation* ρ *of* G*, there is a pair* (J, λ)*, unique up to* G*-conjugacy, such that* λ *embeds in the restriction of* ρ *to* J.

(2) *If* ρ *is an irreducible representation of* G *containing a pair* (J, λ)*, then it is cuspidal.*

(3) *If two irreducible cuspidal representations* ρ, ρ' *both contain* (J, λ)*, then there is an unramified character* $\chi : \mathrm{G} \to \mathrm{R}^\times$ *(that is, trivial on all compact subgroups of* G*) such that* $\rho' \simeq \rho\chi$.

(4) *Given* (J, λ)*, any irreducible subquotient of* $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}}(\lambda)$*, the compact induction of* λ *to* G*, is isomorphic to a quotient of* $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}}(\lambda)$.

(5) *Given* (J, λ)*, the representation* λ *extends to its* G*-normalizer* $\widehat{\mathrm{J}}$*; compact induction from* $\widehat{\mathrm{J}}$ *to* G *induces a bijection between the set of representations of* $\widehat{\mathrm{J}}$ *extending* λ *and that of isomorphism classes of irreducible cuspidal representations of* G *containing* λ.

Example 2.3. — Write $\mathrm{GL}_n(q)$ for the group of invertible $n \times n$ matrices with entries in the residue field of F, and let σ be an irreducible cuspidal representation of $\mathrm{GL}_n(q)$. Inflate σ into an irreducible representation of $\mathrm{K} = \mathrm{GL}_n(\mathcal{O})$ that is trivial on $\mathrm{K}_1 = 1 + \mathscr{M}_n(\mathfrak{p})$, still denoted σ. Then the pairs of the form (K, σ) obtained this way fullfil all the properties 1 to 5 for irreducible cuspidal representations of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ having nonzero K_1-fixed vectors (such irreducible representations are said to have level 0).

Moreover, if $\widehat{\sigma}$ is a representation of KZ (where Z is the centre of G) extending σ and if ρ is the representation of G compactly induced form $\widehat{\sigma}$, then the space ρ^{K_1} of K_1-fixed vectors of ρ – which is naturally a representation of $\mathrm{K}/\mathrm{K}_1 \simeq \mathrm{GL}_n(q)$ – is isomorphic to σ.

Moreover, if ρ is an irreducible representation of G containing (K, σ), then it is supercuspidal if and only if σ is supercuspidal as a representation of $\mathrm{GL}_n(q)$.

Example 2.4. — We give a positive level example for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. Let us fix a uniformizer ϖ of F, a character $\psi : \mathrm{F} \to \mathrm{R}^\times$ trivial on $\mathfrak{p}$ but not on $\mathcal{O}$, and a character $\omega : \mathrm{F}^\times \to \mathrm{R}^\times$ trivial on $1 + \mathfrak{p}$. Define a compact open subgroup:

$$\mathrm{I}_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathcal{O} \\ \mathfrak{p} & \cdots & \cdots & \mathfrak{p} \end{pmatrix}$$

of G. Given $t \in \mathcal{O}^\times$, define a character $\theta = \theta_t$ of I_1 by:

$$\theta(1 + x) = \psi(x_{1,2} + \cdots + x_{n-1,n} + t\varpi^{-1}x_{n,1}), \quad 1 + x \in \mathrm{I}_1.$$

Write $\mathrm{J} = \mathcal{O}^\times \mathrm{I}_1$ and let $\lambda = \lambda_t$ be the character of J defined by $\lambda(xg) = \omega(x)\theta(g)$ for all $x \in \mathcal{O}^\times$ and $g \in \mathrm{I}_1$. The F-algebra E generated by the element:

$$\beta = \begin{pmatrix} 0 & t \cdot \mathrm{id}_{n-1} \\ \varpi & 0 \end{pmatrix} \in \mathscr{M}_n(\mathrm{F})$$

is a totally ramified extension of degree n and uniformizer β. The group $\mathrm{E}^\times$ normalizes the pair (J, λ), and the G-normalizer of the latter is $\widehat{\mathrm{J}} = \mathrm{E}^\times\mathrm{J}$. Given $z \in \mathrm{R}^\times$, there is a unique character ${}_z\lambda = {}_z\lambda_t$ of $\widehat{\mathrm{J}}$ extending λ such that:

$${}_z\lambda(\beta) = z$$

and its compact induction ${}_z\rho = {}_z\rho_t$ is a supercuspidal irreducible representation (of level $1/n$) of G. It is obtained from ${}_1\rho$ by twisting by the unramified character $g \mapsto z^{\mathrm{val}(\det(g))}$, where val denotes the valuation of F giving value 1 to any uniformizer.

Two pairs (J, λ_t), (J, λ_u) with $t, u \in \mathcal{O}^\times$ are G-conjugate if and only if $tu^{-1} \in 1 + \mathfrak{p}$.

Remark 2.5. — This construction shows that, given ℓ and $n \geqslant 1$, there is an irreducible mod ℓ supercuspidal representation of $\mathrm{GL}_n(\mathrm{F})$. Note that it may happen that all level 0 mod ℓ cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ are non-supercuspidal (this is the case, for instance, if $n = q = 2$ and $\ell = 3$).

The pairs (J, λ) appearing in Theorem 2.2 a called the *maximal simple types* of G. The proof of Theorem 2.1 requires a larger family of types, called the *semisimple types* of G (see [10, 23] for a precise definition). We will only give the following crucial fact about these types.

Let $\alpha = (n_1, \dots, n_r)$ be a family of positive integers of sum n. For $i \in \{1, \dots, r\}$, let $(\mathrm{J}_i, \lambda_i)$ be a maximal simple type of $\mathrm{GL}_{n_i}(\mathrm{F})$. Then there exists a semisimple type $(\mathbf{J}, \boldsymbol{\lambda})$ of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ such that the compact induction $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ is isomorphic to the parabolic induction:

$$\mathrm{ind}_{\mathrm{J}_1}^{\mathrm{GL}_{n_1}(\mathrm{F})}(\lambda_1) \times \dots \times \mathrm{ind}_{\mathrm{J}_r}^{\mathrm{GL}_{n_r}(\mathrm{F})}(\lambda_r). \tag{2.1}$$

For instance, if $m_1 = \dots = m_r = 1$ and if λ_i is the trivial character of $\mathrm{J}_i = \mathcal{O}^\times$ for all i, then one can choose for $\boldsymbol{\lambda}$ the trivial character of the standard Iwahori subgroup I of G.

Let us sketch the proof of Theorem 2.1 for the group $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. Let (π, V) be an irreducible representation of G.

Step 1. — We first reduce to the case where π is cuspidal, by applying an appropriate parabolic restriction functor $\boldsymbol{r}_\alpha$ (such that $\boldsymbol{r}_\alpha(\pi)$ is cuspidal) and by using the Geometric Lemma 1.5.

Step 2. — We prove uniqueness up to inertia: there are a unique positive integer r dividing n and an irreducible supercuspidal representation ρ of $\mathrm{GL}_{n/r}(\mathrm{F})$ such that any pair in $\mathrm{scusp}(\pi, \mathrm{V})$ is G-conjugate to a pair of the form:

$$(\mathrm{GL}_{n/r}(\mathrm{F}) \times \dots \times \mathrm{GL}_{n/r}(\mathrm{F}), \rho\chi_1 \otimes \dots \otimes \rho\chi_r)$$

where the χ_i's are unramified characters of $\mathrm{GL}_{n/r}(\mathrm{F})$. This step requires type theory and uniqueness of supercuspidal support for irreducible representations of the groups $\mathrm{GL}_m(q^u)$, where mu divides n.

Step 3. — We finally prove that any pair in $\mathrm{scusp}(\pi, \mathrm{V})$ is G-conjugate to a pair of the form:

$$(\mathrm{GL}_{n/r}(\mathrm{F}) \times \dots \times \mathrm{GL}_{n/r}(\mathrm{F}), \rho \otimes \rho|\det| \otimes \dots \otimes \rho|\det|^{r-1})$$

where $x \mapsto |x|$ denotes the absolute value of F (normalized so that $|\varpi| = q^{-1}$) and ρ is uniquely determined up to a twist by some power of $|\det|$ (note that $\rho|\det|^r$ is isomorphic to ρ). This step requires the theory of Whittaker models [27, III.1], which allows one to prove the following crucial fact: if $\chi_1, \dots, \chi_r$ are unramified characters of $\mathrm{GL}_{n/r}(\mathrm{F})$ such that $\rho\chi_1 \times \dots \times \rho\chi_r$ has a cuspidal subquotient, there is an $i \in \{1, \dots, r\}$ such that, for all $k \in \mathbf{Z}$, there is a $j \in \{1, \dots, r\}$ such that $\rho\chi_i|\det|^k \simeq \rho\chi_j$ (see [22, §8.2]).

This proof uses strong results that are known so far for $\mathrm{GL}_n(\mathrm{F})$ and its inner forms only. They are of two kinds: (a) results from the theory of types, and (b) results from the representation theory of finite reductive groups.

We give more details on Step 2 in the case where π is a level 0 cuspidal representation.

We first introduce a useful tool. Given a smooth R-representation (σ, W) of G, let us form the space $\mathrm{W}^{\mathrm{K}_1}$ of K_1-fixed vectors of W. Since K_1 is normal in $\mathrm{K} = \mathrm{GL}_n(\mathcal{O})$, there is a representation of K on $\mathrm{W}^{\mathrm{K}_1}$, and K_1 acts trivially. This gives us a representation of the quotient K/K_1, which naturally identifies with $\mathrm{GL}_n(q)$. This defines an exact functor from $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ to the category of R-representations of $\mathrm{GL}_n(q)$.

Let $\mathrm{P}=\mathrm{MN}$ be a standard parabolic subgroup (that is, containing all upper triangular matrices) of G. We write $\mathrm{M}(q)$ for the standard Levi subgroup of $\mathrm{GL}_n(q)$ that corresponds to M. By restricting functions from G to K, one can see from the Iwasawa decomposition $\mathrm{G}=\mathrm{PK}$ that the functor $\mathrm{W}\mapsto\mathrm{W}^{\mathrm{K}_1}$ transforms the parabolic induction functor $\boldsymbol{i}_{\mathrm{M,P}}^{\mathrm{G}}$ into the Harish-Chandra induction functor from $\mathrm{M}(q)$ to $\mathrm{GL}_n(q)$, denoted $\mathbf{R}$.

Assume that π occurs as an irreducible subquotient of $\rho_1\times\dots\times\rho_r$ where ρ_i is an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(\mathrm{F})$, and with $n_1+\dots+n_r=n$. Then π^{K_1}, which is nonzero, is a subquotient of:

$$(\rho_1\times\dots\times\rho_r)^{\mathrm{K}_1}\simeq\mathbf{R}\big(\rho_1^{\mathrm{K}_1(n_1)}\otimes\dots\otimes\rho_r^{\mathrm{K}_1(n_r)}\big)$$

where $\mathrm{K}_1(n)$ stands for the K_1-group of $\mathrm{GL}_n(\mathrm{F})$, $n\geqslant 1$. This implies that all the ρ_i's have level 0. Following Example 2.3, write:

$$\pi\simeq\mathrm{ind}_{\mathrm{KF}^\times}^{\mathrm{G}}(\widehat{\sigma}),\quad \rho_i\simeq\mathrm{ind}_{\mathrm{K}(n_i)\mathrm{F}^\times}^{\mathrm{GL}_{n_i}(\mathrm{F})}(\widehat{\sigma}_i),\ \ i\in\{1,2,\dots,r\},$$

where $\mathrm{K}(n)=\mathrm{GL}_n(\mathcal{O})$, $n\geqslant 1$. By taking the K_1-fixed vectors, we get that σ is a subquotient of the Harish-Chandra induction of $\sigma_1\otimes\dots\otimes\sigma_r$. This implies (see Marc Cabanes's lectures) that $\sigma_1\simeq\sigma_2\simeq\dots\simeq\sigma_r$ (denoted σ_0) and $n_1=n_2=\dots=n_r$ (denoted n_0). We get the result by choosing $r=n/n_0$ and $\rho=\rho_1$.

Remark 2.6. — The level 0 case is easy because it is the smallest possible level for an irreducible representation, and because the compatibility of the functor of K_1-fixed vectors with parabolic induction follows from the Iwasawa decomposition. The positive level case is more difficult, and requires the use of endo-classes [8, 6].

2.2. Decomposition of $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$

A (super)cuspidal pair of G is a pair (M,ρ) made of a Levi subgroup $\mathrm{M}\subseteq\mathrm{G}$ and an irreducible (super)cuspidal representation ρ of M.

Definition 2.7. — Two cuspidal pairs (M,ρ) and (M',ρ') in G are *inertially equivalent* if there is an unramified character χ of M such that (M',ρ') is G-conjugate to $(\mathrm{M},\rho\chi)$.

Let $\mathcal{S}(\mathrm{G})$ denote the set of all inertial classes of supercuspidal pairs of G. Let us fix $\Omega\in\mathcal{S}(\mathrm{G})$ and choose $(\mathrm{M},\rho)\in\Omega$ with:

$$\begin{aligned}\mathrm{M}&=\mathrm{GL}_{n_1}(\mathrm{F})\times\dots\times\mathrm{GL}_{n_r}(\mathrm{F}),\\ \rho&=\rho_1\otimes\dots\otimes\rho_r,\end{aligned}$$

where ρ_i is a supercuspidal irreducible representation of $\mathrm{GL}_{n_i}(\mathrm{F})$. For each $i=1,\dots,r$ choose a pair (J_i,λ_i) for ρ_i as in Theorem 2.2 and write:

$$\mathscr{U}(\Omega)=\mathrm{ind}_{\mathrm{J}_1}^{\mathrm{GL}_{n_1}(\mathrm{F})}(\lambda_1)\times\dots\times\mathrm{ind}_{\mathrm{J}_r}^{\mathrm{GL}_{n_r}(\mathrm{F})}(\lambda_r).$$

According to (2.1), this representation can be described as the compact induction of a semisimple type. For instance, if $n_1=\dots=n_r=1$ and all ρ_i are the trivial character of $\mathrm{F}^\times$, then $\mathscr{U}(\Omega)$ is the compact induction $\mathrm{ind}_{\mathrm{I}}^{\mathrm{G}}(1)$ of the trivial character of an Iwahori subgroup I of G.

Theorem 2.8 ([24]). — (1) *For all irreducible representations* (π, V) *of* G *and all* $\Omega \in \mathcal{S}(\mathrm{G})$*, one has:*

$$\mathrm{scusp}(\pi, \mathrm{V}) \in \Omega \quad \Leftrightarrow \quad \pi \text{ is a subquotient of } \mathscr{U}(\Omega).$$

(2) *For all smooth representations* (π, V) *of* G *and all* $\Omega \in \mathcal{S}(\mathrm{G})$*, let* $\mathrm{V}(\Omega)$ *denote the maximal subrepresentation of* V *all of whose irreducible subquotients have supercuspidal support in* Ω*. Then:*

$$\mathrm{V} = \bigoplus_{\Omega \in \mathcal{S}(\mathrm{G})} \mathrm{V}(\Omega).$$

(3) *If* (π, V),(σ, W) *are smooth representations of* G*, then* $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}, \mathrm{W})$ *decomposes canonically as the product of all* $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}(\Omega), \mathrm{W}(\Omega))$*'s for* $\Omega \in \mathcal{S}(\mathrm{G})$*.*

(4) *The full subcategory* $\mathscr{R}_{\mathrm{R}}(\Omega)$ *made of all smooth representations* (π, V) *of* G *such that* $\mathrm{V} = \mathrm{V}(\Omega)$ *is indecomposable.*

The strategy of the proof is very different from Bernstein's proof for complex representations. It uses type theory as well as a decomposition theorem with respect to the supercuspidal support for representations of $\mathrm{GL}_n(q)$.

Example 2.9. — Let (π, V) be a smooth level zero R-representation of G, that is, V is generated by $\mathrm{V}^{\mathrm{K}_1}$. As a representation of $\mathrm{GL}_n(q)$, $\mathrm{V}^{\mathrm{K}_1}$ decomposes as a direct sum:

$$\bigoplus_{[\mathrm{L}, \sigma]} \mathrm{V}^{\mathrm{K}_1}([\mathrm{L}, \sigma])$$

where $[\mathrm{L}, \sigma]$ ranges over all possible supercuspidal supports of $\mathrm{GL}_n(q)$ and where $\mathrm{V}^{\mathrm{K}_1}([\mathrm{L}, \sigma])$ is the maximal subrepresentation of $\mathrm{V}^{\mathrm{K}_1}$ all of whose irreducible subquotients have supercuspidal support $[\mathrm{L}, \sigma]$. Write $\mathrm{V}[\mathrm{L}, \sigma]$ for the subrepresentation of V generated by $\mathrm{V}^{\mathrm{K}_1}([\mathrm{L}, \sigma])$. Then V decomposes as the direct sum of the $\mathrm{V}[\mathrm{L}, \sigma]$'s. Now write:

$$\begin{aligned} \mathrm{L} &= \mathrm{GL}_{n_1}(q) \times \cdots \times \mathrm{GL}_{n_r}(q), \\ \sigma &= \sigma_1 \otimes \cdots \otimes \sigma_r, \end{aligned}$$

where σ_i is a supercuspidal irreducible representation of $\mathrm{GL}_{n_i}(q)$. For each $i = 1, \ldots, r$, inflate σ_i to an irreducible representation of $\mathrm{J}_i = \mathrm{K}(n_i) = \mathrm{GL}_{n_i}(\mathcal{O})$, still denoted σ_i. By Example 2.3, the pair (J_i, σ_i) is a level 0 maximal simple type. Choose a supercuspidal irreducible representation ρ_i of $\mathrm{GL}_{n_i}(\mathrm{F})$ containing the pair (J_i, σ_i). The representation $\rho = \rho_1 \otimes \cdots \otimes \rho_r$ is a supercuspidal irreducible representation of a standard Levi subgroup M of G. Let Ω be the inertial class of the supercuspidal pair (M, ρ). The process $[\mathrm{L}, \sigma] \mapsto \Omega$ is well defined and induces a bijection between supercuspidal supports of $\mathrm{GL}_n(q)$ and inertial class of level 0 supercuspidal pairs of G. Moreover, one has $\mathrm{V}(\Omega) = \mathrm{V}[\mathrm{L}, \sigma]$.

By type theory, one can prove that the endomorphism algebra $\mathrm{End}_{\mathrm{G}}(\mathscr{U}(\Omega))$ is a finite tensor product of affine Hecke algebras of type A. Together with part 2 of Theorem 2.8, this implies that $\mathscr{R}_{\mathrm{R}}(\Omega)$ is indecomposable.

Problem 3: What is the structure of $\mathscr{R}_{\mathrm{R}}(\Omega)$ for $\Omega \in \mathcal{S}(\mathrm{G})$? Find a progenerator P_Ω in $\mathscr{R}_{\mathrm{R}}(\Omega)$ and compute $\mathrm{End}_{\mathrm{G}}(\mathrm{P}_\Omega)$.

For $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, type theory provides candidates for the P_Ω's.

Theorem 2.10 ([14]). — *Let ρ be an irreducible supercuspidal representation of* $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. *Let Ω be its inertial class. Write $n(\rho)$ for the number of unramified characters χ of* G *such that $\rho\chi \simeq \rho$ and v for the ℓ-adic valuation of $q^{n(\rho)} - 1$. There is a progenerator P_Ω of $\mathscr{R}_\mathrm{R}(\Omega)$, with:*

$$\mathrm{End}_\mathrm{G}(\mathrm{P}_\Omega) \simeq \mathrm{R}[\mathrm{X}, \mathrm{X}^{-1}, \mathrm{T}]/(\mathrm{T}^{\ell^v}).$$

When Ω has level 0, Guiraud [15] proved that there is a progenerator P_Ω, and that the computation of its endomorphism algebra reduces to the case where $n_1 = \cdots = n_r$ and $\rho_1 = \cdots = \rho_r$ (with the notation of the beginning of the paragraph).

Lecture 3.

In this lecture, our goal is the classification of all irreducible representations of $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$ having given (super)cuspidal support. For complex representations, this has been done by Zelevinski [32] for $\mathrm{GL}_n(\mathrm{F})$, and by Tadić [26] for its inner forms.

3.1. Reduction to the (super)unipotent case

Assume $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ for simplicity. The indecomposable block corresponding to the inertial class Ω_1 of the pair $(\mathrm{F}^{\times n}, 1_{\mathrm{F}^\times}^{\otimes n})$ is called the *unipotent block*.

An irreducible representation (π, V) of G is *unipotent* if it has supercuspidal support in Ω_1, that is, if π is a subquotient of $\mathscr{U}(\Omega_1) \simeq \mathrm{ind}_\mathrm{I}^\mathrm{G}(1)$ where I denotes the standard Iwahori subgroup of G.

An irreducible representation (π, V) of G is *superunipotent* if $\mathrm{cusp}(\pi) \in \Omega_1$, that is, if V^I is nonzero.

Theorem 3.1 ([23]). — *Let $\mathfrak{s}$ be a cuspidal support in* G. *It writes uniquely as $\mathfrak{s}_1 + \cdots + \mathfrak{s}_t$ such that* (a) *$\mathfrak{s}_j, \mathfrak{s}_k$ have inertially equivalent terms in common if and only if $j = k$, and* (b) *any two terms in $\mathfrak{s}_j$ are inertially equivalent, for all j. Then:*

(1) *The map $(\pi_1, \ldots, \pi_t) \to \pi_1 \times \cdots \times \pi_t$ induces a bijection:*

$$\mathrm{cusp}^{-1}(\mathfrak{s}_1) \times \cdots \times \mathrm{cusp}^{-1}(\mathfrak{s}_t) \to \mathrm{cusp}^{-1}(\mathfrak{s}).$$

(2) *Assume that the terms of $\mathfrak{s}$ are supercuspidal. Then the same map induces a bijection:*

$$\mathrm{scusp}^{-1}(\mathfrak{s}_1) \times \cdots \times \mathrm{scusp}^{-1}(\mathfrak{s}_t) \to \mathrm{scusp}^{-1}(\mathfrak{s}).$$

We are thus reduced to describe $\mathrm{cusp}^{-1}(\Omega_\rho)$ and $\mathrm{scusp}^{-1}(\Omega_\rho)$ where Ω_ρ is the inertial class of $(\mathrm{GL}_{n/r}(\mathrm{F})^r, \rho^{\otimes r})$ and ρ a (super)cuspidal irreducible representation of $\mathrm{GL}_{n/r}(\mathrm{F})$, r dividing n.

Theorem 3.2 ([23]). — *Let ρ be a cuspidal irreducible representation of* $\mathrm{GL}_{n/r}(\mathrm{F})$.

(1) *There are a finite extension* F'/F *of degree dividing n and a bijective map from* $\mathrm{cusp}^{-1}(\Omega_\rho)$ *to the set of superunipotent representations of* $\mathrm{GL}_r(\mathrm{F}')$.

(2) *Assume ρ is supercuspidal. Then there is a bijective map from* $\mathrm{scusp}^{-1}(\Omega_\rho)$ *to the set of unipotent representations of* $\mathrm{GL}_r(\mathrm{F}')$.

Moreover, these bijections preserve the (super)cuspidal support in the following sense. Given an unramified character χ of G, it writes $\chi = \xi \circ \det$ where ξ is an unramified character of $\mathrm{F}^\times$. Then write χ' for the unramified character $\xi' \circ \det$ of $\mathrm{G}' = \mathrm{GL}_r(\mathrm{F}')$, where ξ' is the unramified character of $\mathrm{F}'^\times$ whose value at a uniformizer of F' is the same as that of ξ at a uniformizer of F. We get a bijective map $\chi \mapsto \chi'$ between unramified characters of G and unramified characters of G'. Then an irreducible representation $\pi \in \mathrm{cusp}^{-1}(\Omega_\rho)$ has cuspidal support $\rho\chi_1 + \cdots + \rho\chi_r$ if and only if it corresponds to a superunipotent representation π' of G' having cuspidal support $\chi'_1 + \cdots + \chi'_r$. There is a similar statement for the supercuspidal support in the case where ρ is supercuspidal.

Therefore we are reduced to classify (super)unipotent representations of $\mathrm{GL}_n(F)$.

3.2. Classification of (super)unipotent representations, I

We write $\mathcal{H}_R(G, I)$ for the Hecke-Iwahori R-algebra, that is the space of functions $f : G \to R$ with compact support and such that $f(xgx') = f(g)$ for all $g \in G$, $x, x' \in I$, endowed with the convolution product with respect to the Haar measure on G giving measure 1 to I.

For all smooth representation (π, V), the space:

$$V^I \simeq \mathrm{Hom}_G(\mathcal{C}_c^\infty(I\backslash G, R), V)$$

is made into a right module over the Iwahori-Hecke algebra $\mathcal{H}_R(G, I)$ by the formula:

$$v * f = \sum f(g)\pi(g^{-1})v$$

for $v \in V^I$ and $f \in \mathcal{H}_R(G, I)$, where g ranges over a set of representatives of $I\backslash G$ in G.

If R has characteristic ℓ different from $0, p$, the functor $V \mapsto V^I$ from smooth representations of G to right modules over $\mathcal{H}_R(G, I)$ need not be exact (more precisely, this functor is exact if and only if ℓ does not divide $q-1$). Thus $\mathcal{C}_c^\infty(I\backslash G, R)$ need not be projective as a representation of G, but it has the following crucial property.

Theorem 3.3 ([28]). — (1) *The representation* $\mathcal{C}_c^\infty(I\backslash G, R)$ *is quasi-projective, that is, for any surjective* G*-homomorphism* $\mathcal{C}_c^\infty(I\backslash G, R) \to V$*, the restriction* $\mathcal{H}_R(G, I) \to V^I$ *is also surjective.*

(2) *The functor* $V \to V^I$ *induces a bijection between the isomorphism classes of superunipotent representations of* G *and the isomorphism classes of simple right modules over* $\mathcal{H}_R(G, I)$*.*

But this functor kills all unipotent non-superunipotent representations. In order to deal with these representations as well as the superunipotent ones, Vignéras has introduced the following affine Schur algebra. Write:

$$V = \bigoplus_{\mathcal{P}} \mathcal{C}_c^\infty(\mathcal{P}\backslash G, R)$$

where $\mathcal{P}$ ranges over all standard parahoric subgroups (that is, $I \subseteq \mathcal{P} \subseteq K$) and their conjugates by the G-normalizer of I.

The endomorphism algebra $\mathcal{S}_R(G, I) = \mathrm{End}_G(V)$ is called the affine Schur algebra (see [30]).

Fix a Haar measure μ on G and let $\mathcal{H}_R(G, \mu)$ denote the space $\mathcal{C}_c^\infty(G, R)$ endowed with the convolution product with respect to μ. Any smooth R-representation (π, V) is given a structure of left $\mathcal{H}_R(G, \mu)$-module by:

$$f \cdot v = \int_G f(g)\pi(g)v \, d\mu(g).$$

Let $\mathscr{J} = \mathscr{J}_R(G)$ be the ideal of $\mathcal{H}_R(G, \mu)$ that annihilates the representation $\mathcal{C}_c^\infty(I\backslash G, R)$.

Theorem 3.4 ([30]). — (1) *There is a functor from* $\mathscr{R}_R(G)$ *to the category of right* $\mathcal{S}_R(G, I)$*-modules inducing an equivalence between the full subcategory of* $\mathscr{R}_R(\Omega_1)$ *made of all representations that are killed by* $\mathscr{J}$ *and the category of right* $\mathcal{S}_R(G, I)$*-modules.*

(2) *This induces a bijection between the isomorphism classes of unipotent representations of* G *and the isomorphism classes of simple modules over* $\mathcal{S}_{\mathrm{R}}(\mathrm{G},\mathrm{I})$.

(3) *There is an integer* $\mathrm{N} \geqslant 1$ *such that* $\mathscr{J}^{\mathrm{N}}$ *annihilates the whole block* $\mathscr{R}_{\mathrm{R}}(\Omega_1)$.

3.3. Classification of (super)unipotent representations, II

We now classify unipotent representations of $\mathrm{GL}_n(\mathrm{F})$ by multisegments.

Definition 3.5. — (1) A *segment* is a pair $(\xi, n) \in \mathrm{R}^\times \times \mathbf{Z}_{>0}$.

(2) A *multisegment* is a formal finite sum of segments.

(3) The integer n is called the *length* of the segment (ξ, n), and the length of a multisegment is the sum of the lengths of its segments.

Given a segment (ξ, n), write $\mathrm{Z}(\xi, n)$ for the character:

$$g \mapsto \xi^{-\mathrm{val}(\det(g))}$$

of the group $\mathrm{GL}_n(\mathrm{F})$. For a multisegment $\mathfrak{m} = (\xi_1, n_1) + \cdots + (\xi_r, n_r)$ of length $n \geqslant 1$, we want to define an irreducible subquotient $\mathrm{Z}(\mathfrak{m})$ of:

$$\mathrm{I}(\mathfrak{m}) = \mathrm{Z}(\xi_1, n_1) \times \cdots \times \mathrm{Z}(\xi_r, n_r).$$

Let U be the subgroup of upper triangular unipotent matrices of $\mathrm{GL}_n(\mathrm{F})$. Let us fix a smooth nontrivial character ψ_{F} of F, and set $\psi(u) = \psi_{\mathrm{F}}(u_{1,2} + \cdots + u_{n-1,n})$ for all $u \in \mathrm{U}$.

Definition 3.6. — An irreducible representation (π, V) of G is *generic* if the space $\mathrm{Hom}_{\mathrm{U}}(\pi, \psi)$ is nonzero. (This notion does not depend on the choice of ψ_{F}.)

Given a multisegment $\mathfrak{m}$ of length $n \geqslant 1$ as above, write $\mu_{\mathfrak{m}}$ for the partition of n conjugate to $(n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r)$.

Proposition 3.7. — (1) $\mu_{\mathfrak{m}}$ *is the unique maximal partition (for the dominance order) of* n *such that the parabolic restriction* $\boldsymbol{r}_{\mu_{\mathfrak{m}}}(\mathrm{I}(\mathfrak{m}))$ *contains a generic irreducible subquotient.*

(2) *Such a generic irreducible subquotient is unique, and occurs with multiplicity* 1.

Write $\mathrm{Z}(\mathfrak{m})$ for the unique irreducible subquotient of $\mathrm{I}(\mathfrak{m})$ such that the parabolic restriction $\boldsymbol{r}_{\mu_{\mathfrak{m}}}(\mathrm{Z}(\mathfrak{m}))$ contains a generic irreducible subquotient.

Theorem 3.8 ([22]). — *The map* $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ *is a bijection between multisegments of length* n *and isomorphism classes of unipotent representations of* $\mathrm{GL}_n(\mathrm{F})$.

This bijection does not depend on the choice of $\sqrt{q} \in \mathrm{R}^\times$.

Proof. — For injectivity, $\mu_{\mathfrak{m}}$ can be recovered by the uniqueness property in Proposition 3.7(1). Then $\mathfrak{m}$ can be recovered by looking at the generic irreducible subquotient in $\boldsymbol{r}_{\mu_{\mathfrak{m}}}(\mathrm{Z}(\mathfrak{m}))$.

For surjectivity, one uses a counting argument that is based on the classification of all simple modules over $\mathcal{H}_{\mathrm{R}}(\mathrm{G},\mathrm{I})$ by aperiodic multisegments [1, 12]. □

Now write ℓ for the characteristic of R and define an integer $e \geqslant 0$ by:

$$e = \begin{cases} 0 & \text{if } \ell = 0, \\ \text{the smallest } k \geqslant 2 \text{ such that } 1 + q + \cdots + q^{k-1} = 0 & \text{if } \ell > 0. \end{cases}$$

Then the multisegment $\mathfrak{m}$ writes uniquely as:

$$\mathfrak{m} = \mathfrak{a} + \sum_{u\geqslant 0}\sum_{n\geqslant 1}\sum_{\xi} c(\xi,n,u)\cdot(\xi,n)_u$$

where ξ ranges over a set of representatives of $\mathrm{R}^\times/q^{\mathbf{Z}}$ in $\mathrm{R}^\times$ and:

(1) $(\xi,n)_u$ is the multisegment:

$$(\xi,n)+(\xi q,n)+\cdots+(\xi q^{e\ell^u-1},n)$$

where e is as above, and $c(\xi,n,u)$ is an integer in $\{0,1,\dots,\ell-1\}$;

(2) $\mathfrak{a}$ is q-aperiodic, which means that for all segments (ζ,d) and all $v\geqslant 0$, the multisegment $(\zeta,d)_v$ does not occur in $\mathfrak{a}$ (when $\ell=0$, any multisegment is q-aperiodic thus $\mathfrak{a}=\mathfrak{m}$);

Given an integer $u\geqslant 0$, the induced representation $1\times\nu\times\cdots\times\nu^{e\ell^u-1}$ (where ν is the absolute value $x\mapsto|x|$) possesses a unique cuspidal irreducible subquotient, denoted ρ_u. Given $\xi\in\mathrm{R}^\times$, we also write $\chi_{\xi,u}$ for the unramified character $\mathrm{Z}(\xi,e\ell^u)$ of $\mathrm{GL}_{e\ell^u}(\mathrm{F})$.

Theorem 3.9 ([22]). — *The cuspidal support of* $\mathrm{Z}(\mathfrak{m})$ *is:*

$$\mathrm{scusp}(\mathrm{Z}(\mathfrak{a}))+\sum_{u\geqslant 0}\sum_{n\geqslant 1}\sum_{\xi} n\cdot c(\xi,n,u)\cdot\rho_u\chi_{\xi,u}.$$

We finally have the following decomposition theorem.

Theorem 3.10 ([22]). — *Let* $\mathfrak{m}$ *be a multisegment. Then the semisimplification of* $\mathrm{I}(\mathfrak{m})$ *writes:*

$$\mathrm{Z}(\mathfrak{m})+\sum_{\mathfrak{n}} d_{\mathfrak{m},\mathfrak{n}}\cdot\mathrm{Z}(\mathfrak{n})$$

where $\mathfrak{n}$ *ranges over all multisegments and* $d_{\mathfrak{m},\mathfrak{n}}\in\mathbf{Z}_{\geqslant 0}$*, with the following property: if* $d_{\mathfrak{m},\mathfrak{n}}\neq 0$*, then* $\mu_{\mathfrak{n}}$ *is smaller than* $\mu_{\mathfrak{m}}$ *(for the dominance order).*

Remark 3.11. — When G is a non-split inner form of $\mathrm{GL}_n(\mathrm{F})$, there is no theory of generic representations for G. In order to define the irreducible representation $\mathrm{Z}(\mathfrak{m})$ of G, we introduce the notion of residually generic representation (see [22]) by using the functor $\mathrm{W}\mapsto\mathrm{W}^{\mathrm{K}_1}$ defined in Paragraph 2.1 (and more general functors coming from type theory to deal with positive level representations).

3.4. Comments

When R is the field of complex numbers, the map $\mathfrak{m}\mapsto\mathrm{Z}(\mathfrak{m})$ gives Zelevinski's classification of all irreducible representations having nonzero Iwahori-fixed vectors. When the segments of $\mathfrak{m}$ are put in a suitable order, the representation $\mathrm{Z}(\mathfrak{m})$ can be characterized as the unique irreducible subrepresentation of $\mathrm{I}(\mathfrak{m})$.

There is also a Langlands classification $\mathfrak{m}\mapsto\mathrm{L}(\mathfrak{m})$ where $\mathrm{L}(\mathfrak{m})$ is uniquely determined as the unique irreducible quotient of $\mathrm{J}(\mathfrak{m})=\mathrm{L}(\xi_1,n_1)\times\cdots\times\mathrm{L}(\xi_r,n_r)$ when the segments are put in a suitable order, and where $\mathrm{L}(\xi,n)$ is the unique generic irreducible representation with the same cuspidal support as $\mathrm{Z}(\xi,n)$. These two classifications are exchanged by the Zelevinski involution.

When R has nonzero characteristic $\ell\neq p$, it is also possible to define a Langlands classification $\mathfrak{m}\mapsto\mathrm{L}(\mathfrak{m})$ and a mod ℓ Zelevinski involution that exchanges Z and L. However $\mathrm{L}(\xi,n)$ need not be generic anymore.

Lecture 4.

In this section, G is the group $\mathrm{GL}_n(\mathrm{F})$.

4.1. Reduction mod ℓ

Let us fix a prime number $\ell \neq p$ and an algebraic closure $\overline{\mathbf{Q}}_\ell$ of the field of ℓ-adic numbers. The residue field $\overline{\mathbf{F}}_\ell$ of its ring of integers $\overline{\mathbf{Z}}_\ell$ is an algebraic closure of a finite field of characteristic ℓ.

Definition 4.1. — An irreducible $\overline{\mathbf{Q}}_\ell$-representation (π, V) of G is said to be *integral* if V has a G-stable lattice L, that is a free $\overline{\mathbf{Z}}_\ell$-module generated by a $\overline{\mathbf{Q}}_\ell$-basis of V.

Then $\mathrm{L} \otimes \overline{\mathbf{F}}_\ell$ is a smooth $\overline{\mathbf{F}}_\ell$-representation of finite length of G, and its semisimplification does not depend on the choice of L; it is denoted $\mathbf{r}_\ell(\pi)$ (see [31] and [27, II.5.11]).

We fix a square root of q in $\overline{\mathbb{Z}}_\ell^\times \subseteq \overline{\mathbb{Q}}_\ell^\times$. By reducing mod the maximal ideal of $\overline{\mathbf{Z}}_\ell$, it gives us a square root of q in $\overline{\mathbb{F}}_\ell^\times$.

We write MS(R) for the semigroup of multisegments made of segments (ξ, n) with $\xi \in \mathrm{R}^\times$ and Z_R for the bijection from MS(R) to the set of isomorphism classes of unipotent representations.

A multisegment $\mathfrak{m} = (\xi_1, n_1) + \cdots + (\xi_r, n_r) \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$ is *integral* if the ξ_i's belong to $\overline{\mathbb{Z}}_\ell^\times$. If $\mathfrak{m}$ as above is integral, define:

$$\mathbf{r}_\ell(\mathfrak{m}) = (\overline{\xi}_1, n_1) + \cdots + (\overline{\xi}_r, n_r) \in \mathrm{MS}(\overline{\mathbf{F}}_\ell)$$

where $\overline{\xi}$ denotes the image of $\xi \in \overline{\mathbb{Z}}_\ell^\times$ in $\overline{\mathbb{F}}_\ell^\times$.

Lemma 4.2. — (1) *Let* $\mathfrak{m} \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$. *Then* $\mathrm{Z}(\mathfrak{m})$ *is integral if and only if* $\mathfrak{m}$ *is integral.*

(2) *Let* π *be an integral unipotent* $\overline{\mathbf{Q}}_\ell$*-representation of* G. *Then all irreducible subquotients of* $\mathbf{r}_\ell(\pi)$ *are unipotent.*

Proof. — For 2, let $\mathfrak{m} = (\xi_1, n_1) + \cdots + (\xi_r, n_r) \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$ be an integral multisegment such that π is isomorphic to $\mathrm{Z}(\mathfrak{m})$. By construction, π is an irreducible subquotient of:

$$\mathrm{I}(\mathfrak{m}) = \mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\xi_1, n_1) \times \cdots \times \mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\xi_r, n_r).$$

Since reduction mod ℓ commutes with parabolic induction, all irreducible subquotients of $\mathbf{r}_\ell(\pi)$ are subquotients of:

$$\mathbf{r}_\ell(\mathrm{I}(\mathfrak{m})) = \mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\overline{\xi}_1, n_1) \times \cdots \times \mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\overline{\xi}_r, n_r) = \mathrm{I}(\mathbf{r}_\ell(\mathfrak{m})).$$

The result follows. □

Theorem 4.3. — *Let* $\mathfrak{m} \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$ *be an integral multisegment. Then:*

$$\mathbf{r}_\ell(\mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\mathfrak{m})) = \mathrm{Z}_{\overline{\mathbf{F}}_\ell}(\mathbf{r}_\ell(\mathfrak{m})) + \sum_{\mathfrak{n}} a(\mathfrak{m}, \mathfrak{n}) \cdot \mathrm{Z}_{\overline{\mathbf{F}}_\ell}(\mathfrak{n})$$

where $\mathfrak{n}$ *ranges over all multisegments and* $a(\mathfrak{m}, \mathfrak{n}) \in \mathbf{Z}_{\geqslant 0}$, *with the property: if* $a(\mathfrak{m}, \mathfrak{n})$ *is nonzero, then* $\mu_\mathfrak{n}$ *is smaller than* $\mu_\mathfrak{m}$ *(for the dominance order).*

Example 4.4. — Assume $\mathrm{G} = \mathrm{GL}_2(\mathbf{Q}_5)$ and $\ell = 3$ (see Example 1.11). Thus we have $q = 5$.

The unipotent $\overline{\mathbf{Q}}_\ell$-representation corresponding to $(1, 2) \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$ is the trivial $\overline{\mathbf{Q}}_\ell$-character of G, whose reduction mod ℓ is the trivial $\overline{\mathbf{F}}_\ell$-character of G, that corresponds to the multisegment $(\overline{1}, 2) \in \mathrm{MS}(\overline{\mathbf{F}}_\ell)$.

Now consider $\mathfrak{m} = (1, 1) + (q, 1) \in \mathrm{MS}(\overline{\mathbf{Q}}_\ell)$. We write St for the Steinberg $\overline{\mathbf{Q}}_\ell$-representation of G and π for the cuspidal subquotient of the $\overline{\mathbf{F}}_\ell$-representation V of Example 1.11. We have the following diagram:

$$\begin{array}{ccc} (1,1)+(q,1) & \xrightarrow{\mathrm{Z}_{\overline{\mathbf{Q}}_\ell}} & \mathrm{St}\mathrm{Z}_{\overline{\mathbf{Q}}_\ell}(\sqrt{q},2) \\ \downarrow{\scriptstyle \mathbf{r}_\ell} & & \downarrow{\scriptstyle \mathbf{r}_\ell} \\ (1,1)+(-1,1) & \xrightarrow[\mathrm{Z}_{\overline{\mathbf{F}}_\ell}]{} & \pi + \mathrm{Z}_{\overline{\mathbf{F}}_\ell}(-1,2) \end{array}$$

and $\pi = \mathrm{Z}_{\overline{\mathbf{F}}_\ell}((1,1) + (-1,1))$ is unipotent but not superunipotent.

4.2. The local Langlands correspondence

Let us fix a separable closure $\overline{\mathrm{F}}$ of F. Its residue field $\overline{\mathfrak{k}}$ is a separable closure of the residue field $\mathfrak{k}$ of F. The Galois group $\Gamma_\mathrm{F} = \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ acts on $\overline{\mathfrak{k}}$ such that there is a surjective group homomorphism:

$$\gamma_\mathrm{F} : \Gamma_\mathrm{F} \to \mathrm{Gal}(\overline{\mathfrak{k}}/\mathfrak{k}).$$

The group Γ_F is profinite, and $\mathrm{I}_\mathrm{F} = \mathrm{Ker}(\gamma_\mathrm{F})$ is a closed subgroup. Let $\mathrm{Frob}_\mathrm{F} \in \mathrm{Gal}(\overline{\mathfrak{k}}/\mathfrak{k})$ be the Frobenius automorphism $x \mapsto x^q$. Then write:

$$\mathrm{W}_\mathrm{F} = \{g \in \Gamma_\mathrm{F} \mid \gamma_\mathrm{F}(g) \in \mathrm{Frob}_\mathrm{F}^{\mathbf{Z}}\}.$$

It is called the Weil group of F. There is a unique topology on W_F such that:

(1) the topology of I_F induced by W_F and that induced by Γ_F coincide;
(2) the subgroup I_F is open in W_F.

Note that this is not the topology on W_F induced by Γ_F (for which (2) is not satisfied).

Remark 4.5. — The group W_F is locally compact, and its element 1 has a basis of neighborhoods made of compact open pro-p-subgroups. There is a notion of smooth R-representation of W_F, just as for the group G. There is also a notion of reduction mod ℓ for (integral) finite-dimensional $\overline{\mathbf{Q}}_\ell$-representations of W_F.

Now write:

$$\begin{array}{rcl} \mathscr{G}_n^0(\mathrm{F}) & = & \{\text{isomorphism classes of } n\text{-dimensional irreducible } \mathbf{C}\text{-representations of } \mathrm{W}_\mathrm{F}\}, \\ \mathscr{A}_n^0(\mathrm{F}) & = & \{\text{isomorphism classes of irreducible cuspidal } \mathbf{C}\text{-representations of } \mathrm{GL}_n(\mathrm{F})\}. \end{array}$$

The local Langlands correspondence [20, 17, 18] asserts that there exists a unique family of bijections:

$$\pi_n^0 : \mathscr{G}_n^0(\mathrm{F}) \to \mathscr{A}_n^0(\mathrm{F}), \quad n \geqslant 1$$

satisfying certain specific conditions that we do not give explicitely here.

Now choose an isomorphism $\alpha : \mathbf{C} \to \overline{\mathbf{Q}}_\ell$. By extension of scalars, any smooth complex representation of $\mathrm{W}_\mathrm{F}, \mathrm{G}$ gives rise to a smooth $\overline{\mathbf{Q}}_\ell$-representation of $\mathrm{W}_\mathrm{F}, \mathrm{G}$. It also gives bijections:

$${}_\alpha\pi_n^0 : \mathscr{G}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell) \to \mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell), \quad n \geqslant 1$$

depending on α, because π_n^0 for n even depends on the choice of $\sqrt{q} \in \mathbf{C}^\times$.

Theorem 4.6 **([29])**. — (1) *For any $\rho \in \mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)$, the reduction $\mathbf{r}_\ell(\rho)$ is irreducible and cuspidal.*

(2) *The map:*

$$\mathbf{r}_\ell : \mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)^{\mathrm{int}} \to \mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{F}}_\ell)$$

is surjective, where $\mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)^{\mathrm{int}}$ *denotes the subset of integral representations in* $\mathscr{A}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)$.

(3) *A representation* $\sigma \in \mathscr{G}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)$ *is integral if and only if* ${}_\alpha\pi_n^0(\sigma)$ *is integral.*

(4) *Assume* $\sigma, \sigma' \in \mathscr{G}_n^0(\mathrm{F}, \overline{\mathbf{Q}}_\ell)$ *are integral. Then:*

$$\mathbf{r}_\ell(\sigma) = \mathbf{r}_\ell(\sigma') \quad \Leftrightarrow \quad \mathbf{r}_\ell({}_\alpha\pi_n^0(\sigma)) = \mathbf{r}_\ell({}_\alpha\pi_n^0(\sigma')).$$

(5) $\mathbf{r}_\ell(\sigma)$ *is irreducible if and only if* $\mathbf{r}_\ell({}_\alpha\pi_n^0(\sigma))$ *is supercuspidal.*

(6) *This induces a bijection* ${}_\alpha\overline{\pi}_n^0$ *between isomorphism classes of irreducible* n*-dimensional* $\overline{\mathbf{F}}_\ell$*-representationsof* W_F *and isomorphism classes of irreducible supercuspidal* $\overline{\mathbf{F}}_\ell$*-representations of* G.

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