The Bernstein decomposition for smooth complex representations of $\mathrm{GL}_n(F)$

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Introduction

Let F be a nonarchimedean locally compact field with residue characteristic denoted p, let $\mathbf{G}$ be a connected reductive group defined over F and let R be an algebraically closed field. We write G for the group of F-points of $\mathbf{G}$ and $\mathcal{R}_R(G)$ for the category of all smooth representations of G on R-vector spaces. From a representation theoretic point of view, it is natural to study the structure of this category. In the 1980's, Bernstein [2] proved the following result when R is the field $\mathbb{C}$ of complex numbers.

Theorem. *The category $\mathcal{R}_{\mathbb{C}}(G)$ decomposes into a product of indecomposable summands, called blocks.*

The purpose of these notes is to prove this theorem in the case where $\mathbf{G}$ is GL_n for $n \geq 1$. We will define the notion of inertial class for the group $\mathrm{GL}_n(F)$ and attach to each irreducible smooth representation of $\mathrm{GL}_n(F)$ an inertial class. To each inertial class Ω we then attach a full subcategory of $\mathcal{R}_{\mathbb{C}}(\mathrm{GL}_n(F))$, denoted $\mathcal{R}_{\mathbb{C}}(\Omega)$, made of the representations of $\mathrm{GL}_n(F)$ all of whose irreducible subquotients have inertial class Ω. We then prove that $\mathcal{R}_{\mathbb{C}}(\mathrm{GL}_n(F))$ decomposes into the product of all the $\mathcal{R}_{\mathbb{C}}(\Omega)$'s (see Theorem 3.7). We do not prove that these subcategories are indecomposable, nor investigate their precise structure. The main references are [3, 19, 20].

In the last section, we say a few words about the case where R is an algebraic closure $\overline{\mathbb{F}}_l$ of a finite field of characteristic a prime number l.

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Preliminaries

All representations of topological groups that we will consider in these notes will be supposed to be smooth representations on R-vector spaces. Moreover, apart from Section 4, R will be the field of complex numbers. For definitions and basic facts about the theory of smooth representations of locally profinite groups, we will refer to Blondel's notes in the present volume.

1 Compact representations

For a locally profinite group with centre Z, Blondel has introduced the notion of Z-compact representation, that is a smooth representation all of whose coefficients are compactly supported modulo Z. In order to avoid the centre Z, which will be a source of trouble for our purpose, we will consider compact representations (see Definition 1.1 below) on groups with compact centre. Of course, $\mathrm{GL}_n(F)$ has centre isomorphic to $F^\times$, which is not compact. We will see in Section 2 how to deal with this.

1.1 The decomposition theorem

From now on, and until the end of Section 1, H will denote a locally profinite, unimodular group with compact centre. We assume that, for any compact open subgroup K of H, the set H/K is countable. Then Schur's lemma holds (Blondel, Lemma 3.22).

We choose once and for all a Haar measure dh on H and write $\mathcal{H}(H)$ for the Hecke algebra of H with respect to this Haar measure (Blondel, §2.3).

We remind the reader that a *coefficient* of a smooth representation (π, V) of H is a function from H to $\mathbb{C}$ of the form:

$$c_{v,\xi} : h \mapsto \xi(\pi(h)v)$$

for some $v \in V$ and $\xi \in \tilde{V}$, where $\tilde{V}$ denotes the space of smooth linear forms on V. This defines a map, denoted c, from $V \otimes \tilde{V}$ to the space of smooth complex functions on H.

Definition 1.1. *A compact representation of H is a smooth representation of H all of whose coefficients are compactly supported.*

The space generated by all the coefficients of a smooth representation (π, V) is denoted by $\mathcal{C}(\pi)$. Thus (π, V) is compact if and only if $\mathcal{C}(\pi)$ is a subspace of $\mathcal{H}(H)$.

We have the following important property (Blondel, Corollary 3.26).

Proposition 1.2. *Any irreducible smooth compact representation of H is admissible.*

Compact representations are of interest to us because of the following theorem.

Theorem 1.3. *Let (τ, W) be an irreducible compact representation of H. Given any smooth representation (π, V) of H, there is a unique pair (V^τ, V_τ) of subspaces of V such that:*

1. *V^τ and V_τ are stable by H, and $V = V^\tau \oplus V_\tau$;*
2. *V^τ is a direct sum of copies of (τ, W);*
3. *no irreducible subquotient of V_τ is isomorphic to (τ, W).*

Our first goal is to prove this theorem.

1.2 Formal degree of an irreducible compact representation

The Hecke algebra $\mathcal{H}(H)$ is equipped with an action of $H \times H$ by:

$$(h, h') \cdot f : x \mapsto f(h^{-1}xh'), \quad f \in \mathcal{H}(H), \quad h, h', x \in H. \tag{1}$$

This makes $\mathcal{H}(H)$ into a smooth representation of $H \times H$. Given (π, V) a smooth representation of H, we can do the same for the endomorphism algebra $\mathrm{End}_{\mathbb{C}}(V)$ by setting:

$$(h, h') \cdot \phi : v \mapsto \pi(h)\phi(\pi(h'^{-1})v), \quad \phi \in \mathrm{End}_{\mathbb{C}}(V), \quad h, h' \in H, \quad v \in V,$$

but $\mathrm{End}_{\mathbb{C}}(V)$ as a representation of $H \times H$ is not smooth in general. When (π, V) is irreducible and admissible, one can compute the smooth part $\mathrm{End}_{\mathbb{C}}(V)^{\infty}$ of $\mathrm{End}_{\mathbb{C}}(V)$.

Lemma 1.4. *Let (π, V) be an admissible irreducible smooth representation of H. The linear map:*

$$\alpha : V \otimes \tilde{V} \to \mathrm{End}_{\mathbb{C}}(V)^{\infty} \subseteq \mathrm{End}_{\mathbb{C}}(V)$$

defined by:

$$\alpha(w \otimes \xi) : v \mapsto \xi(v)w, \quad w, v \in V, \quad \xi \in \tilde{V},$$

is an isomorphism of representations of $H \times H$.

Proof. The map α is clearly injective and $H \times H$-equivariant. We are going to prove surjectivity by "approximations". This means that we will prove that, for any compact open subgroup K of H, the map:

$$\alpha^K : (V \otimes \tilde{V})^{K \times K} \to \mathrm{End}_{\mathbb{C}}(V)^{K \times K}$$

obtained by restricting α is surjective. Since (π, V) is smooth, and since $\mathrm{End}_{\mathbb{C}}(V)^{\infty}$ is the union of all $\mathrm{End}_{\mathbb{C}}(V)^{K \times K}$ when K varies, the lemma will follow.

Since (π, V) is admissible, the space V^K is finite-dimentional. The space $V^K \otimes \tilde{V}^K$ embeds naturally in $(V \otimes \tilde{V})^{K \times K}$, and the canonical pairing $V^K \otimes \tilde{V}^K \to R$ identifies $\tilde{V}^K$ with the dual of V^K. We thus have:

$$\dim(V \otimes \tilde{V})^{K \times K} \geq \dim(V^K \otimes \tilde{V}^K) = (\dim V^K)^2.$$

Given $\phi \in \mathrm{End}_{\mathbb{C}}(V)$, we have $\phi \in \mathrm{End}_{\mathbb{C}}(V)^{K \times K}$ if and only if:

$$\phi(\pi(k)w) = \pi(k')\phi(w), \quad k, k' \in K, \quad w \in V.$$

As K is compact, V decomposes into the direct sum $V^K \oplus V(K)$ of K-stable subspaces, where $V(K)$ is the subspace generated by the vectors of the form $\pi(k)w - w$ for $k \in K$ and $w \in V$ (Blondel, §2.7). Thus any $\phi \in \mathrm{End}_{\mathbb{C}}(V)^{K \times K}$ induces an endomorphism ϕ^K of V^K, and the map $\phi \mapsto \phi^K$ is injective because the restriction of ϕ to $V(K)$ is zero. Thus we have:

$$\dim \mathrm{End}_{\mathbb{C}}(V)^{K \times K} \leq \dim \mathrm{End}_{\mathbb{C}}(V^K) = (\dim V^K)^2$$

which ends the proof of the lemma. □

Now let (τ, W) be a compact irreducible representation. By Proposition 1.2, it is admissible, thus Lemma 1.4 applies. Given $f \in \mathcal{H}(H)$, write $\tau(f)$ for the endomorphism of W defined by:

$$\tau(f)(w) = \int_H f(h)\tau(h)w \, dh, \quad w \in W,$$

where dh is the Haar measure on H that we have chosen when we defined $\mathcal{H}(H)$. Let us consider the $H \times H$-equivariant linear map:

$$\begin{array}{rcl} \mathcal{H}(H) & \to & \mathrm{End}_{\mathbb{C}}(W)^{\infty} \\ f & \mapsto & \tau(f) \end{array}$$

which we still denote τ. The map c from $W \otimes \tilde{W}$ to $\mathcal{H}(H)$ associated with the representation (τ, W) is not $H \times H$-equivariant. We introduce a map c' from $W \otimes \tilde{W}$ to $\mathcal{H}(H)$ defined by:

$$c'_{w,\xi} : h \mapsto c_{w,\xi}(h^{-1}), \quad w \in W, \quad \xi \in \tilde{W}, \quad h \in H.$$

Then c' is $H \times H$-equivariant and we have the following diagram:

$$\begin{array}{ccc} W \otimes \tilde{W} & \xrightarrow{\;c'\;} & \mathcal{H}(H) \\ & \underset{\alpha}{\searrow} & \downarrow \tau \\ & & \mathrm{End}_{\mathbb{C}}(W)^{\infty} \end{array}$$

where all the maps are $H \times H$-equivariant, thus $\gamma = \tau \circ c' \circ \alpha^{-1}$ is an $H \times H$-endomorphism of $\mathrm{End}_{\mathbb{C}}(W)^{\infty}$. As $\mathrm{End}_{\mathbb{C}}(W)^{\infty}$ is an irreducible representation of $H \times H$, Schur's lemma implies that γ is a scalar, denoted $d(\tau)$ and called the *formal degree* of (τ, W).

Remark 1.5.

1. *The scalar $d(\tau)$ depends on the choice of the Haar measure chosen on H, but it is of no importance for our purpose.*
2. *When H has finite order and the measure gives measure 1 to H, any irreducible representation (τ, W) is compact and $d(\tau) = \dim_{\mathbb{C}}(W)^{-1}$.*

The only thing we will need to know about the formal degree is the following property.

Proposition 1.6. *We have $d(\tau) \neq 0$.*

Proof. For this, we will need the following lemma. For a proof, see for instance [19] or [3].

Lemma 1.7. *Let $f \in \mathcal{H}(H)$ be nonzero. There is an irreducible representation (π, V) of H such that $\pi(f) \neq 0$.*

We apply this lemma as follows. Let $f \in \mathcal{C}(\tau) \subseteq \mathcal{H}(H)$ be nonzero. The lemma gives us an irreducible representation (π, V) of H such that $\pi(f) \neq 0$. We are going to prove that this π is isomorphic to τ. Indeed, fix a vector $v \in V$ and define the map:

$$\lambda : W \otimes \tilde{W} \to V$$

given by $\lambda(w \otimes \xi) = \pi(c'_{w,\xi})(v)$ for all $w \in W$, $\xi \in \tilde{W}$. We make H act on the W-factor (that is, $\tilde{W}$ is just a multiplicity space and $W \otimes \tilde{W}$ is a direct sum of copies of W). As V is irreducible and λ is linear and H-equivariant, λ is either zero or surjective. If λ is zero for all $v \in V$, then the restriction of π to $\mathcal{C}(\tau)$ is zero. In particular, we have $\pi(f) = 0$: contradiction. Thus there is a $v \in V$ such that λ is surjective. As $W \otimes \tilde{W}$ is a direct sum of copies of W, so is V, thus V and W are isomorphic.

We thus have $\tau(f) \neq 0$. But $f \in \mathcal{C}(\tau)$ has the form $f = c' \circ \alpha^{-1}(\phi)$ for some $\phi \in \mathrm{End}_{\mathbb{C}}(W)^{\infty}$. Thus:

$$0 \neq \tau(f) = d(\tau) \cdot \phi.$$

This ends the proof of Proposition 1.6. □

1.3 Proof of Theorem 1.3

Given a smooth representation (π, V) of H, we have to attach two subspaces V^{τ} and V_{τ} to it, satisfying the properties of Theorem 1.3.

When H is compact, (π, V) is semisimple. Thus the space V^{τ} is the τ-isotypic component of V and V_{τ} is the unique complement of V^{τ} in V. Moreover, there is an idempotent $e^{\tau} \in \mathcal{H}(H)$ such that $\pi(e^{\tau})V = V^{\tau}$ and $\pi(1 - e^{\tau})V = V_{\tau}$.

When H is not compact, such an idempotent e^{τ} does not always exists in the Hecke algebra $\mathcal{H}(H)$. We proceed by approximations: given K a compact open subgroup of H, we will define an idempotent $e_K^{\tau} \in \mathcal{H}(H)$ with good properties, and then define V^{τ} and V_{τ} by passing to the limit $K \to \{1\}$.

Given K a compact open subgroup of H, write 1_K for the characteristic function of K on H and $\mathrm{meas}(K)$ for the measure of K given by the Haar measure on H that has been chosen.

Proposition 1.8. *Let K be a compact open subgroup of H, and write:*

$$e_K = \frac{1_K}{\mathrm{meas}(K)} \in \mathcal{H}(H)$$

for the idempotent associated with it. There is a unique $e_K^{\tau} \in \mathcal{H}(H)$ such that, for any irreducible smooth representation (σ, E) of H, one has:

$$\sigma(e_K^{\tau}) = \begin{cases} \tau(e_K) & \text{if } \sigma \text{ and } \tau \text{ are isomorphic,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Unicity follows from Lemma 1.7. For the existence, set:

$$e_K^{\tau} = \frac{1}{d(\tau)} \cdot c' \circ \alpha^{-1} \circ \tau(e_K), \tag{2}$$

which is well defined by Proposition 1.6. By definition of the formal degree, it satisfies $\tau(e_K^{\tau}) = \tau(e_K)$. Moreover, the proof of Proposition 1.6 shows that the image of $\mathcal{C}(\tau)$ by σ is zero for all σ nonisomorphic to τ, thus $\sigma(e_K^{\tau}) = 0$. □

Proposition 1.9. *Write $*$ for the convolution product in $\mathcal{H}(H)$. We have the following properties.*

1. *Assume $K \supseteq K'$ are compact open subgroups of H. Then:*

$$e^\tau_{K'} * e^\tau_K = e_K * e^\tau_{K'} = e^\tau_{K'} * e_K = e^\tau_K.$$

2. *For all $h \in H$, one has $(h,h) \cdot e^\tau_K = e^\tau_{hKh^{-1}}$ (see (1) for the notation).*

Proof. We prove that $e^\tau_{K'} * e^\tau_K = e^\tau_K$. By Lemma 1.7, it suffices to prove $\sigma(e^\tau_{K'} * e^\tau_K) = \sigma(e^\tau_K)$ for all irreducible representation (σ, E) of H. We have:

$$\sigma(e^\tau_{K'} * e^\tau_K) = \sigma(e^\tau_{K'})\sigma(e^\tau_K).$$

If σ is not isomorphic to τ, this is zero as well as $\sigma(e^\tau_K)$. If σ is isomorphic to τ, we get:

$$\sigma(e^\tau_{K'} * e^\tau_K) = \tau(e_{K'})\tau(e_K) = \tau(e_K) = \sigma(e^\tau_K).$$

The other equalities follow by using a similar proof. □

In particular, Property 1 implies that e^τ_K is an idempotent in $\mathcal{H}(H)$. Define:

$$V^\tau = \sum_K \text{Im } \pi(e^\tau_K), \quad V_\tau = \bigcap_K \text{Ker } \pi(e^\tau_K),$$

where the sum and the intersection are taken over all compact open subgroups K of H. We now prove Theorem 1.3. We first check that the subspaces V^τ and V_τ are H-stable. Given $v \in V_\tau$, one has $\pi(e^\tau_K)v = 0$ for all compact open subgroups K of H. For $g \in H$, one has:

$$\pi(e^\tau_K)\pi(g)v = \pi(g)\pi(e^\tau_{g^{-1}Kg})v = 0.$$

Given $v \in V^\tau$, there is a compact open subgroup K of H and $w \in V$ such that $v = \pi(e^\tau_K)w$. Thus:

$$\pi(g)v = \pi(e^\tau_{gKg^{-1}})\pi(g)w \in \text{Im } \pi(e^\tau_{gKg^{-1}}) \subseteq V^\tau.$$

We now check that V decomposes into the direct sum of V^τ and V_τ. Given $v \in V^\tau \cap V_\tau$, we have $v = \pi(e^\tau_K)w$ for some K and $w \in V$. Then:

$$0 = \pi(e^\tau_K)v = \pi(e^\tau_K)\pi(e^\tau_K)w = \pi(e^\tau_K)w = v.$$

Now let $v \in V$. As V is smooth, there is a compact open subgroup K of H such that $v \in V^K = \pi(e_K)V$, that is $\pi(e_K)v = v$. Now set $v_1 = \pi(e^\tau_K)v \in V^\tau$ and $v_0 = v - v_1$. It remains to prove that v_0 lies in V_τ. Given $K' \subseteq K$, one has:

$$\pi(e^\tau_{K'})v_0 = \pi(e^\tau_{K'})v - \pi(e^\tau_{K'})\pi(e^\tau_K)v = \pi(e^\tau_{K'} * e_K)v - \pi(e^\tau_{K'} * e^\tau_K)v$$

which is zero by Proposition 1.9. We now check that V^τ is a direct sum of copies of W. We first remind the reader that the map:

$$c : W \otimes \tilde{W} \to \mathcal{H}(H)$$

is injective, since it is nonzero and $W \otimes \tilde{W}$ is irreducible as a representation of $H \times H$. As a representation of H on the first factor, it is a direct sum of copies of W. The image $\text{Im}(c) = \mathcal{C}(\tau)$

is thus a direct sum of copies of W. Now for any compact open subgroup K, one has $e_K^\tau \in \mathcal{C}(\tau)$, thus $\mathcal{H}(H)e_K^\tau$, which is the sub-$\mathcal{H}(H)$-module of $\mathcal{C}(\tau)$ generated by e_K^τ, is a subrepresentation of $\mathcal{C}(\tau)$, and thus is a direct sum of copies of W. For any $v \in V$, the subrepresentation $\pi(\mathcal{H}(H)e_K^\tau)v$ of V is a quotient of $\mathcal{H}(H)e_K^\tau$, thus is a direct sum of copies of W. When K and v vary, the sum of all $\pi(\mathcal{H}(H)e_K^\tau)v$ is V^τ, which therefore is a direct sum of copies of W.

Assume that there are H-stable subspaces $V_1 \subseteq V_2 \subseteq V_\tau$ such that V_2/V_1 is isomorphic to W as a representation of H. For any compact open subgroup K, one has a sequence:

$$0 \to \pi(e_K^\tau)V_1 \to \pi(e_K^\tau)V_2 \to \pi(e_K^\tau)(V_2/V_1) \to 0$$

of $\mathbb{C}$-vector spaces, which is exact because $\pi(e_K^\tau)$ is an idempotent. Now remark that:

$$\pi(e_K^\tau)(V_2/V_1) \simeq \tau(e_K^\tau)W = W^K.$$

But $\pi(e_K^\tau)V_1 = \pi(e_K^\tau)V_2 = \{0\}$ and $W^K \neq \{0\}$ for K small enough: contradiction.

We now finish the proof of Theorem 1.3 by proving uniqueness of the decomposition. Write $V = X \oplus Y$ where X and Y are H-stable subspaces of V, no irreducible subquotient of Y is isomorphic to W and X is a direct sum of copies of W. For all compact open subgroups K in H, we have $\pi(e_K^\tau)(Y) = 0$ and $\pi(e_K^\tau)(X) = X^K$. Therefore, $Y \subseteq V_\tau$ and $X \subseteq V^\tau$. From the first inclusion, we get $X \simeq V/Y = V^\tau \oplus (V_\tau/Y)$, thus $Y = V_\tau$. Then it follows that $X = V^\tau$.

1.4 The compact part of a smooth representation of H

If (τ_i, W_i), $i = 1, \ldots, r$ are nonisomorphic irreducible compact representations of H, then any smooth representation (π, V) of H decomposes into:

$$V = V^{\tau_1} \oplus V^{\tau_2} \oplus \cdots \oplus V^{\tau_r} \oplus V_{\tau_1, \tau_2, \ldots, \tau_r}$$

where V^{τ_i} is a direct sum of copies of W_i and where $V_{\tau_1,\tau_2,\ldots,\tau_r}$ is the intersection of all the V_{τ_i}'s, $i = 1, \ldots, r$, and therefore has no irreducible subquotient isomorphic to any (τ_i, W_i).

One must be careful when dealing with infinitely many nonisomorphic irreducible compact representations of H. Let $\mathrm{Cpt}(H)$ be the set of all isomorphism classes of irreducible compact representations of H and set:

$$V_{\mathrm{c}} = \bigoplus_{(\tau,W)\in\mathrm{Cpt}(H)} V^\tau, \quad V_{\mathrm{nc}} = \bigcap_{(\tau,W)\in\mathrm{Cpt}(H)} V_\tau.$$

Therefore V_{c} and V_{nc} are subrepresentations of V. Note that no irreducible subquotient of V_{nc} is compact. One has a map:

$$V_{\mathrm{c}} \oplus V_{\mathrm{nc}} \to V$$

which is injective, but may not be surjective. Let us introduce the following condition on the group H:

(∗) *for any compact open subgroup K of H, there are finitely many irreducible compact representations (τ, W) of H such that $W^K \neq \{0\}$.*

We now have the following theorem.

Theorem 1.10. *The decomposition $V = V_{\mathrm{c}} \oplus V_{\mathrm{nc}}$ holds for any smooth representation (π, V) of H if and only if H satisfies $(*)$.*

Proof. Assume that H satisfies $(*)$. Let $v \in V$ and let K be a compact open subgroup of H such that $v \in V^K$. Then there are finitely many (τ, W) in $\mathrm{Cpt}(H)$ such that W^K is nonzero, that is, such that $(V^\tau)^K$ is nonzero. Therefore the sum:

$$v_{\mathrm{c}} = \sum_{(\tau,W)\in\mathrm{Cpt}(H)} \pi(e_K^\tau)v$$

is well defined and belongs to V_{c}. This defines a map:

$$\theta : V \to V_{\mathrm{c}}$$

which is zero on V_{nc} and restricts to the identity map on V_{c}. Thus θ is a projection onto V_{c}, with kernel containing V_{nc}. Now write $v_{\mathrm{nc}} = v - v_{\mathrm{c}}$. We want $v_{\mathrm{nc}} \in V_{\mathrm{nc}}$, that is:

$$\pi(e_K^\tau)v_{\mathrm{nc}} = 0$$

for all compact open subgroups $K \subseteq H$ and $(\tau, W) \in \mathrm{Cpt}(H)$. Write:

$$\pi(e_K^\tau)v_{\mathrm{nc}} = \pi(e_K^\tau)v - \pi(e_K^\tau)\left(\sum_{\tau'} \pi(e_K^{\tau'})v\right) = \pi(e_K^\tau)v - \pi(e_K^\tau)\pi(e_K^\tau)v$$

which is 0 because $\pi(e_K^\tau)\pi(e_K^{\tau'}) = 0$ if τ, τ' are non-isomorphic. Therefore V decomposes into the direct sum $V_{\mathrm{c}} \oplus V_{\mathrm{nc}}$. For the converse, see for instance [19]. □

2 The cuspidal part of a smooth representation

Let G be the group $\mathrm{GL}_n(F)$, $n \geq 1$. Write val_F for the valuation on F. The group:

$$H = \{g \in G \mid \mathrm{val}_F(\det(g)) = 0\}$$

satisfies the conditions of Paragraph 1.1, it is normal in G and the quotient G/H is isomorphic to $\mathbb{Z}$ (Blondel, §4.5). Thus Theorem 1.3 holds for smooth representations of H.

The following exercise shows that Theorem 1.3 does not hold when H is replaced by G.

Exercise 2.1. *Assume $n = 1$. Make $G = F^\times$ act on $\mathbb{C}[\mathbb{Z}]$, the group algebra of $\mathbb{Z}$, by:*

$$g \cdot f : x \mapsto f(x + \mathrm{val}_F(g)), \quad g \in G, \quad f \in \mathbb{C}[\mathbb{Z}], \quad x \in \mathbb{Z}.$$

Show that this smooth representation of G does not decompose into a direct sum of characters.

2.1 From compact to cuspidal representations

In this paragraph we investigate the relationship between irreducible compact representations of H and irreducible cuspidal representations of G. In Blondel's course (Theorem 4.38), you have seen two equivalent definitions of *cuspidal*. We will need both of them but, for the moment, an irreducible cuspidal representation of G will be for us an irreducible representation of G whose restriction to H is compact.

Let (ρ, V) be an irreducible cuspidal representation of G. Its restriction to H has an irreducible subquotient (τ, W) which is compact. By Theorem 1.3, the space W is a direct summand of V.

Lemma 2.2. *Let (ρ, V) be an irreducible cuspidal representation of G and (τ, W) be a compact irreducible subquotient of the restriction of ρ to H. Write N for the normalizer of τ in G. There is a representation $(\hat{\tau}, W)$ of N which extends τ and such that ρ is isomorphic to $\mathrm{ind}_N^G(\hat{\tau})$, where ind_N^G denotes compact induction from N to G (Blondel, §1.4).*

Proof. First note that N has finite index in G because it contains both H and the centre Z, and HZ has finite index n in G. Thus the set of isomorphism classes of the τ^g, $g \in G$ is finite. Note that all the τ^g are irreducible compact representations of H.

Write $\rho|_{G'}$ for the restriction of ρ to a subgroup G' of G. We have an injective map i from τ to $\rho|_H$. Write τ_Z for the unique representation of HZ extending τ and having the same central character as ρ (see Blondel, §3.2 for the definition of the central character). Thus i can be extended to an injective map:

$$\tau_Z \to \rho|_{HZ}.$$

Of course the normalizer of τ_Z in G is equal to N. As G/HZ is finite cyclic, there is a $u \in N$ such that the group N is generated by HZ and u. By definition, there is a $A \in \mathrm{Aut}_{\mathbb{C}}(W)$ such that:

$$\tau_Z(uxu^{-1}) = A \circ \tau(x) \circ A^{-1}$$

for all $x \in HZ$. By Schur's lemma, such an A is unique up to a nonzero scalar. Let m be the index of HZ in N, that is, the order of the image of u in N/HZ. Then:

$$\tau_Z(u^m x u^{-m}) = A^m \circ \tau(h) \circ A^{-m} = \tau_Z(u^m) \circ \tau(h) \circ \tau_Z(u^m)^{-1}$$

and it follows from Schur's Lemma again that A^m and $\tau_Z(u^m)$ are equal up to a nonzero scalar. By replacing A by λA for a suitable choice of $\lambda \in \mathbb{C}^\times$, one may, and will, assume that $A^m = \tau_Z(u^m)$. Define a map $t : u^{\mathbb{Z}} \times HZ \to \mathrm{Aut}_{\mathbb{C}}(W)$ by:

$$t(u^k, x) = A^k \tau_Z(x), \quad k \in \mathbb{Z}, \quad x \in HZ.$$

Then t factors through a map $\hat{\tau}_A : N \to \mathrm{Aut}_{\mathbb{C}}(W)$ which is a representation of N extending τ. Now we have:

$$\mathrm{ind}_{HZ}^N(\tau_Z) \simeq \hat{\tau}_A \otimes \mathbb{C}[N/HZ].$$

As N/HZ is cyclic, $\mathbb{C}[N/HZ]$ decomposes into the direct sum of all characters of N trivial on HZ. Thus there is some character $\chi : N \to \mathbb{C}^\times$ trivial on HZ such that there is an injective map from $\hat{\tau}_A\chi$ to $\rho|_N$. Write $\hat{\tau} = \hat{\tau}_A\chi$. By Frobenius reciprocity (Blondel, §1.4), we get a map:

$$\mathrm{ind}_N^G(\hat{\tau}) \to \rho$$

which is surjective since ρ is irreducible. It remains to prove that $\mathrm{ind}_N^G(\hat{\tau})$ is irreducible. Let Y be a nonzero G-stable subspace. Then the restriction $Y|_N$ contains some G-conjugate of $\hat{\tau}$. As it is stable by G, it also contains $\hat{\tau}$. Thus $Y = \mathrm{ind}_N^G(\hat{\tau})$ because $\mathrm{ind}_N^G(\hat{\tau})|_N$ is the direct sum of all the G-conjugates of $\hat{\tau}$. □

Definition 2.3. *1. A character χ of G is said to be unramified if it is trivial on H.*

2. Two irreducible cuspidal representations ρ, ρ' of G are said to be inertially equivalent if there is an unramified character χ of G such that ρ' is isomorphic to $\rho\chi$.

Corollary 2.4. *Let ρ, ρ' be irreducible cuspidal representations of G. The three following conditions are equivalent:*

1. *ρ and ρ' are inertially equivalent.*
2. *$\rho|_H$ and $\rho'|_H$ are isomorphic.*
3. *$\rho|_H$ and $\rho'|_H$ have an irreducible factor in common.*

Proof. We clearly have $1 \Rightarrow 2 \Rightarrow 3$. Now assume that $\rho|_H$ and $\rho'|_H$ have an irreducible factor (τ, W) in common. Thus by Lemma 2.2 we have:

$$\rho \simeq \mathrm{ind}_N^G(\hat{\tau}), \quad \rho' \simeq \mathrm{ind}_N^G(\hat{\tau}'),$$

where N is the normalizer of τ in G and $\hat{\tau}$ and $\hat{\tau}'$ are suitable representations of N extending τ. By the proof of Lemma 2.2 (Schur's lemma) we know that $\hat{\tau}' = \hat{\tau}\chi_N$ for some character χ_N of N trivial on H. This gives us an unramified character χ of G such that $\rho' \simeq \rho\chi$. □

In conclusion, the restriction to H of an irreducible cuspidal representation of G is a finite direct sum of G-conjugates of an irreducible compact representation of H, and this induces a bijection between:

1. inertial classes of irreducible cuspidal representations of G;
2. G-conjugacy classes of irreducible compact representations of H.

2.2 The group H satisfies the finiteness condition $(*)$

We now prove that H satisfies the finiteness condition $(*)$ introduced in Paragraph 1.4. According to Section 2.1, the property $(*)$ for H is equivalent to the finiteness, for any K, of the number of inertial classes of irreducible cuspidal representations of G having nonzero K-fixed vectors. This can be done via the Uniform Admissibility Theorem, asserting that, given any compact open subgroup K of G, there is a constant $c(G, K)$ such that, for any irreducible representation (π, V) of G, one has:

$$\dim(V^K) \leq c(G, K).$$

This can be proven by a careful study of the Hecke algebra $\mathcal{H}(G, K)$ and the use of the Cartan decomposition (Blondel, Proposition 4.36). Moreover, the Uniform Admissibility Theorem holds for any p-adic reductive group. For a proof, see for instance [3, 19].

In our case, that is for $G = \mathrm{GL}_n(F)$, there is another way to prove this, by using the explicit description of irreducible cuspidal representations of G by compact induction (see Bushnell's notes in the present volume). More precisely, there is a family of pairs (J, λ) with J a compact open subgroup of G and λ an irreducible representation of J with the following conditions:

1. for any irreducible cuspidal representation ρ of G, there is (J, λ) such that λ is a subrepresentation of $\rho|_J$, and such a (J, λ) is unique up to G-conjugacy;

2. two irreducible cuspidal representations of G have a pair (J, λ) in common if and only if they are inertially equivalent.

Thus we have a bijection between:

1. inertial classes of irreducible cuspidal representations of G;

2. G-conjugacy classes of irreducible compact representations of H;

3. G-conjugacy classes of pairs (J, λ).

Thus $(*)$ is equivalent to the finiteness, for any K, of the number of G-conjugacy classes of pairs (J, λ) with nonzero $J \cap K$-fixed vectors. This follows from the construction of the pairs (J, λ) in [6] and from the finiteness theorems of [9].

2.3 The cuspidal part of a smooth representation

According to Theorem 1.10, for any smooth representation (π, V) of H, there is a decomposition $V = V_{\mathrm{c}} \oplus V_{\mathrm{nc}}$. Now let (π, V) be a smooth representation of G. By restricting to H, we get:

$$V|_H = V_{\mathrm{c}} \oplus V_{\mathrm{nc}},$$

with V_{c} the direct sum of the V^τ for all τ in $\mathrm{Cpt}(H)$. The subspaces V_{c} and V_{nc} are G-stable because an irreducible representation of H is compact if and only if all its G-conjugates are.

Definition 2.5. *Let V_{cusp} and V_{ind} be the representations of G on V_{c} and V_{nc} respectively.*

The irreducible subquotients of V_{cusp} are cuspidal, and none of V_{ind} is. Write:

$$V_{\mathrm{cusp}}|_H = \bigoplus_{(\tau, W) \in \mathrm{Cpt}(H)} V^\tau.$$

Write $[\tau]$ for the G-conjugacy class of τ (which is finite) and set:

$$V[\tau] = \bigoplus_{\tau' \in [\tau]} V^{\tau'}.$$

Then $V[\tau]$ is G-stable and V_{cusp} is the direct sum of the $V[\tau]$'s. Thus we are reduced to describe $V[\tau]$. As we will see, it is not semisimple in general.

Proposition 2.6. *Let ρ, ρ' be two irreducible subquotients of $V[\tau]$. Then they are cuspidal and inertially equivalent.*

Proof. As $V[\tau]|_H$ is a direct sum of copies of G-conjugates of τ, the representations $\rho|_H$ and $\rho'|_H$ have a factor in common. The proposition then follows from Corollary 2.4. $\square$

In conclusion, we have the following theorem.

Theorem 2.7. *1. Any smooth representation (π, V) of G decomposes into:*

$$V = V_{\text{cusp}} \oplus V_{\text{ind}}$$

where all irreducible subquotients of V_{cusp} are cuspidal and none of V_{ind} is.

2. For Ω an inertial class of irreducible cuspidal representations of G, let $V(\Omega)$ be the maximal subrepresentation of V whose all irreducible subquotients are in Ω. Then:

$$V_{\text{cusp}} = \bigoplus_{\Omega} V(\Omega)$$

where Ω ranges over all inertial class of irreducible cuspidal representations of G.

3 The noncuspidal part of a smooth representation

First we need to generalize the notion of inertial class.

3.1 The cuspidal support of an irreducible representation

Let (π, V) be a smooth irreducible representation of G. There are (Blondel, Proposition 4.35) a parabolic pair (P, L) of G and an irreducible cuspidal representation (σ, W) of L such that π embeds into the induced representation $i_{L,P}^G(\sigma)$. (See Blondel, §4.1 for the notation $i_{L,P}^G, r_{L,P}^G$.)

Remark 3.1. *Remind that "cuspidal" means "Z-compact" or equivalently "with no nonzero proper Jacquet module" (Blondel, Theorem 4.38).*

The pair (L, σ) associated with (π, V) has some uniqueness property.

Lemma 3.2. *Let (P, L), (Q, M) be two parabolic pairs in G and let (σ, W) be a cuspidal representation of L. Write π for the representation $r_{M,Q}^G(i_{L,P}^G(\sigma))$ of M.*

1. *If M does not contain any G-conjugate of L, then π is zero.*
2. *If M is not G-conjugate to L, then π has no irreducible cuspidal subquotient.*
3. *If M, L are G-conjugate, then any irreducible subquotient of π is G-conjugate to a subquotient of σ.*

Proof. For a proof, see [4, Corollary 2.13] or [19, VI.5.3]. $\square$

Corollary 3.3. *1. The pair (L, σ) associated with (π, V) is unique up to G-conjugacy. Its G-conjugacy class is called the cuspidal support of (π, V).*

2. *All the irreducible subquotients of $i_{L,P}^G(\sigma)$ have the same cuspidal support, that is the G-conjugacy class of (L,σ).*

Proof. By adjointness, the Jacquet module $r_{L,P}^G(\pi)$ has an irreducible quotient isomorphic to σ. Assume that we have an embedding of π in $i_{L',P'}^G(\sigma')$ for another triple (P',L',σ'). If we apply the exact functor $r_{L,P}^G$, we get an injective map:

$$r_{L,P}^G(\pi) \to r_{L,P}^G(i_{L',P'}^G(\sigma')).$$

As σ is an irreducible quotient of the left hand side, Lemma 3.2 implies that L and L' are G-conjugate, and even that (L,σ) and (L',σ') are G-conjugate. This proves 1.

Now let π' be an irreducible subquotient of $i_{L,P}^G(\sigma)$. Fix a triple (P',L',σ') associated with π' such that there is an embedding of π' in $i_{L',P'}^G(\sigma')$. Thus $r_{L',P'}^G(\pi')$ has an irreducible quotient isomorphic to σ'. By exactness of $r_{L',P'}^G$, we deduce that $r_{L',P'}^G(i_{L,P}^G(\sigma))$ has an irreducible subquotient isomorphic to σ'. Then Lemma 3.2 implies that (L,σ) and (L',σ') are G-conjugate. This proves 2. □

Remark 3.4. *If $\mathbb{C}$ is replaced by an algebraically closed field R of characteristic not p, then parts 2 of Lemma 3.2 and of Corollary 3.3 do not hold in general (see Example 4.3).*

3.2 The decomposition theorem

We introduce the following definition.

Definition 3.5. 1. *A cuspidal pair of G is a pair (L,σ) made of a Levi subgroup L of G and an irreducible cuspidal representation σ of L.*

2. *Two cuspidal pairs (L,σ) and (L',σ') of G are inertially equivalent in G if there is an unramified character χ of L (i.e. trivial on $L\cap H$) such that $(L,\sigma\chi)$ and (L',σ') are G-conjugate.*

Let $\mathcal{O}(G)$ denote the set of all inertial classes of G, that is, of all inertial equivalence classes of cuspidal pairs of G. Given a smooth representation (π,V) of G and $\Omega\in\mathcal{O}(G)$, write $V(\Omega)$ for the maximal subrepresentation of V all of whose irreducible subquotients have cuspidal support in Ω.

Definition 3.6. *We say the representation (π,V) is split if V decomposes into the direct sum of all the $V(\Omega)$, $\Omega\in\mathcal{O}(G)$.*

Our goal is to prove the following theorem.

Theorem 3.7. 1. *All smooth representations of G are split.*

2. *For any $\Omega\neq\Omega'$ in $\mathcal{O}(G)$, we have $\mathrm{Hom}_G(V(\Omega),V(\Omega'))=\{0\}$.*

Remark 3.8. *Note that Part 2 of the theorem follows from the definition of $V(\Omega)$. Indeed, if there is a nonzero map $f:V(\Omega)\to V(\Omega')$, its image contains an irreducible subquotient whose cuspidal support is in Ω and Ω', which contradicts the fact that $\Omega\neq\Omega'$.*

In order to prove this theorem, we introduce the two following functors. Write A for the set of all finite families $(n_1, \dots, n_r)$ of positive integers with sum n. A parabolic subgroup of G is said to be *standard* if it contains the minimal parabolic subgroup made of upper triangular matrices; a Levi subgroup of G is said to be *standard* if it contains the minimal Levi subgroup made of diagonal matrices (Blondel, §4.2). Standard parabolic and Levi subgroups of G are parametrized by the elements of A. Given $\alpha \in A$, we write r_α and i_α for the functors of parabolic restriction and induction corresponding to the standard parabolic pair (P_α, L_α) of G attached to α.

Given a smooth representation (π, V) of G, set:

$$\mathbf{R}(V) = (r_\alpha(V)_{\mathrm{cusp}})_{\alpha \in A}.$$

Given a family $((L_\alpha, \sigma_\alpha))_{\alpha \in A}$, where σ_α is a smooth cuspidal representation of L_α, set:

$$\mathbf{I}((L_\alpha, \sigma_\alpha)_{\alpha \in A}) = \bigoplus_{\alpha \in A} i_\alpha(\sigma_\alpha).$$

These functors fulfill the following properties.

Lemma 3.9. *1.* $\mathbf{R}$ *is left adjoint to* $\mathbf{I}$*.*

2. $\mathbf{I}$ *and* $\mathbf{R}$ *are exact.*

3. For any smooth representation (π, V) *of* G*, one has* $\mathbf{R}(V) = 0$ *if and only if* $V = 0$ *and the canonical map* $V \to \mathbf{IR}(V)$ *is injective.*

Proof. Properties 1 and 2 are clear, since r_α is left adjoint to i_α and i_α, r_α are both exact. Given a nonzero representation (π, V) of G, there is $\alpha \in A$ such that $r_\alpha(V)$ is nonzero and cuspidal, thus $\mathbf{R}(V)$ is nonzero. Now write η_V for the canonical map $V \to \mathbf{IR}(V)$ and let V' be its kernel. One has the following commutative diagram:

$$\begin{array}{ccc} V' & \xrightarrow{\;i\;} & V \\ {\scriptstyle \eta_{V'}}\downarrow & & \downarrow{\scriptstyle \eta_V} \\ \mathbf{IR}(V') & \xrightarrow[\mathbf{IR}(i)]{} & \mathbf{IR}(V) \end{array}$$

where i is the natural inclusion map of V' in V. Here $\mathbf{IR}(i)$ is injective thanks to Property 2 above. As $\eta_V \circ i = 0$, we get $\mathbf{IR}(i) \circ \eta_{V'} = 0$, which implies $V' = 0$. □

Lemma 3.10. *For any smooth representation* (π, V) *of* G*, the representation* $\mathbf{IR}(V)$ *is split.*

Proof. It is enough to prove that $i_\alpha(\sigma)$ is split for some $\alpha \in A$ and some cuspidal representation (σ, W) of L_α. According to Theorem 2.7 and to the fact that i_α commutes with infinite direct sums, it is enough to prove this when $W = W(\Omega)$ for Ω the inertial class of a cuspidal irreducible representation of L_α. Write Ω_G for the inertial class of G induced by Ω. We are going to prove that all irreductible subquotients of $i_\alpha(\sigma)$ have their cuspidal support in Ω_G, which will prove the lemma.

Let ρ be an irreductible subquotient of $i_\alpha(\sigma)$. By Lemma 3.2, all the irreducible subquotients of $r_\alpha(\rho)$ are isomorphic to a G-conjugate of a subquotient of σ, that is, are in Ω. This implies that ρ has cuspidal support in Ω_G. □

Lemma 3.11. *Let (π, V) be a smooth representation of G and let W be a subrepresentation of V. Assume that V is split. Then W is split.*

Proof. Write V as the direct sum of all the $V(\Omega)$'s for $\Omega \in \mathcal{O}(G)$. It is enough to prove that:

$$W = \bigoplus_{\Omega \in \mathcal{O}(G)} (W \cap V(\Omega))$$

since one clearly has $W \cap V(\Omega) = W(\Omega)$. Write Y for the quotient of W by the right hand side and assume it is nonzero. Pick an irreducible subquotient of Y and write Ω for the inertial class of its cuspidal support. This subquotient is an irreducible subquotient of $W/W \cap V(\Omega)$, thus of $(W + V(\Omega))/V(\Omega)$, thus of $V/V(\Omega)$. Thus it appears in the direct sum of the $V(\Omega')$ for $\Omega' \neq \Omega$: contradiction. ☐

3.3 Further questions

We have defined a partition:

$$\mathrm{Irr}_{\mathbb{C}}(G) = \coprod_{\Omega \in \mathcal{O}(G)} \Omega$$

of the set $\mathrm{Irr}_{\mathbb{C}}(G)$ of all isomorphism classes of irreducible representations of G. This reflects a decomposition:

$$\mathcal{R}_{\mathbb{C}}(G) = \prod_{\Omega \in \mathcal{O}(G)} \mathcal{R}_{\mathbb{C}}(\Omega) \tag{3}$$

of the category of all smooth complex representations of G, where $\mathcal{R}_{\mathbb{C}}(\Omega)$ denotes the full subcategory made of representations V such that $V = V(\Omega)$. Several questions arise.

- **Q1** Are the subcategories $\mathcal{R}_{\mathbb{C}}(\Omega)$ indecomposable? That is, is the decomposition (3) the finest possible decomposition of the category $\mathcal{R}_{\mathbb{C}}(G)$?
- **Q2** How to describe the category $\mathcal{R}_{\mathbb{C}}(\Omega)$ for $\Omega \in \mathcal{O}(G)$?
- **Q3** What if $\mathrm{GL}_n(F)$ is replaced by an arbitrary p-adic reductive group?
- **Q4** What if $\mathbb{C}$ is replaced by any algebraically closed field R of positive characteristic?

We start by Question 3: as has been said in the Introduction, the theorem stated there holds for any p-adic reductive group (see [3]).

The answer to Question 1 is yes (for any p-adic reductive group G). This is not easy to prove. For this, given $\Omega \in \mathcal{O}(G)$, one constructs a representation Π_Ω of $\mathcal{R}_{\mathbb{C}}(\Omega)$ which is projective and finitely generated, and such that all irreducible representations with cuspidal support in Ω are isomorphic to a quotient of Π_Ω. In that case it follows that $\mathcal{R}_{\mathbb{C}}(\Omega)$ is equivalent to the category of modules over $\mathrm{End}_G(\Pi_\Omega)$. This requires the following deep result, known as the second adjointness property. For a proof, see [5].

Theorem 3.12. *Let (P, L) be a parabolic pair in G. Write $\overline{P}$ for the parabolic subgroup of G opposite to P with respect to L. Then $i^G_{L,P}$ has a right adjoint, which is $r^G_{L,\overline{P}}$.*

Question 2 has been partly answered above: $\mathcal{R}_{\mathbb{C}}(\Omega)$ is equivalent to the category of modules over the algebra $\mathrm{End}_G(\Pi_\Omega)$. It is natural to demand a more precise description of this algebra. This is a difficult question, and such a description is known only for some classical groups (see for instance [16]). Bushnell and Kutzko have developed another approach to this question: the theory of types. The strategy is the following:

1. for each inertial class $\Omega \in \mathcal{O}(G)$, construct a type $(J_\Omega, \lambda_\Omega)$, that is a pair made of a compact open subgroup J_Ω of G and an irreducible representation λ_Ω of J_Ω such that, for all irreducible representations (π, V) of G, the restriction of π to J_Ω contains λ_Ω if and only if π has cuspidal support in Ω;

2. if $(J_\Omega, \lambda_\Omega)$ is a type for Ω, then $\mathcal{R}_{\mathbb{C}}(\Omega)$ is equivalent to the category of modules over the intertwining algebra $H_{\mathbb{C}}(\Omega) = \mathrm{End}_G(\mathrm{ind}_{J_\Omega}^G(\lambda_\Omega))$. Therefore it remains to give a description of this algebra by generators and relations.

For $G = \mathrm{GL}_n(F)$, this programme has been carried out by Bushnell and Kutzko [6, 10]. The $H_{\mathbb{C}}(\Omega)$'s are tensor products of Hecke-Iwahori algebras of type A. For $G = \mathrm{SL}_n(F)$, see [7, 8, 14]. For inner forms of $\mathrm{GL}_n(F)$, see P. Broussous, M. Grabitz, V. Sécherre, S. Stevens and E. Zink. For classical groups, see L. Blasco, C. Blondel and S. Stevens.

Example 3.13. *Assume that $G = \mathrm{GL}_2(F)$. Write O_F for the ring of integers of F and P_F for its maximal ideal. Write q_F for the cardinality of the residue field of O_F.*

1. *Assume Ω is the inertial class of an irreducible cuspidal representation ρ of G. See Bushnell's course for the construction of a type $(J_\Omega, \lambda_\Omega)$. Then the algebra $H_{\mathbb{C}}(\Omega)$ is isomorphic to the algebra $\mathbb{C}[T, T^{-1}]$ of Laurent polynomials in one variable T.*

2. *Assume Ω is the inertial class of $(L, \chi_1 \otimes \chi_2)$, where L is the Levi subgroup made of diagonal matrices and where χ_1, χ_2 are characters of $F^\times$.*

   (a) *Assume $\chi_1\chi_2^{-1}$ is unramified on $F^\times$, that is trivial on $O_F^\times$, the group of units in $F^\times$. Then one may assume χ_1 and χ_2 to be trivial. In this case, the subgroup:*

   $$I = \begin{pmatrix} O_F^\times & O_F \\ P_F & O_F^\times \end{pmatrix}$$

   and its trivial character provide a type for Ω. Then $H_{\mathbb{C}}(\Omega)$ is the Iwahori-Hecke algebra of type A_1 and parameter q_F. It has generators S, T with relations:

   - *$(S+1)(S-q_F) = 0$;*
   - *T^2 is invertible and commutes with S.*

   (b) *Assume $\chi_1\chi_2^{-1}$ is ramified. Then there is a type $(J_\Omega, \lambda_\Omega)$ for Ω with the following properties:*

   - *J_Ω is a compact open subgroup of I such that:*

   $$\begin{aligned} J_\Omega &= (J_\Omega \cap N) \cdot (J_\Omega \cap L) \cdot (J_\Omega \cap N^-), \\ J_\Omega \cap L &= O_F^\times \times O_F^\times, \end{aligned}$$

   where N and N^- denote the subgroups of upper and lower triangular unipotent matrices of G, respectively;

- *λ_Ω is trivial on $J_\Omega \cap N$ and $J_\Omega \cap N^-$, and one has:*

$$\lambda_\Omega\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \chi_1(x)\chi_2(y)$$

  for $x, y \in O_F^\times$.

The algebra $H_\mathbb{C}(\Omega)$ is isomorphic to $\mathbb{C}[X, X^{-1}, Y, Y^{-1}]$, where X, Y are two variables.

Finally, for an answer to Question 4, see the next section.

4 Modular smooth representations of $\mathrm{GL}_n(F)$

From now on, G denotes any p-adic reductive group, and we replace $\mathbb{C}$ by the field $\overline{\mathbb{F}}_l$ with l prime. We are interested in the smooth l-modular representations of G, that is smooth representations of G on $\overline{\mathbb{F}}_l$-vector spaces. There is an enormous difference between the case where $l \neq p$ and the case where $l = p$.

4.1 The $l \neq p$ case

In this case, G has a compact open subgroup K such that the index of any open subgroup of K is invertible in $\overline{\mathbb{F}}_l$. Thus there is a Haar measure on G. The theory of l-modular smooth representations of p-adic reductive groups with $l \neq p$ has been developed by M.-F. Vignéras [21]. Thanks to the fact that p is invertible in $\overline{\mathbb{F}}_l$, the following important properties remain valid for (smooth) l-modular representations.

1. Any irreducible l-modular representation of G has a central character and is admissible.
2. Any irreducible l-modular representation of G has a unique cuspidal support, where cuspidal means Z-compact.
3. The parabolic functors are exact and preserve admissibility and finite length.
4. An irreducible representation of G is cuspidal if and only if all its proper Jacquet modules are zero.

But there are also two major phenomenons, which make the theory of smooth l-modular representations of G more difficult than the complex theory:

1. Representations of compact open subgroups of G may not be semi-simple.
2. A cuspidal irreducible representation of G may appear as a subquotient of a proper parabolically induced representation.

This leads to the definition of a supercuspidal representation.

Definition 4.1. *An irreducible representation of G is called supercuspidal if it does not appear as a subquotient of any representation of the form $i_{L,P}^G(\sigma)$, where P is a proper parabolic subgroup of G and σ is an* irreducible *smooth representation of L.*

Remark 4.2. *If we do not require σ to be irreducible in Definition 4.1, we obtain a definition of "supercuspidal" which is a priori stronger. These two definition are expected to be equivalent. This is know for $G = \mathrm{GL}_n(F)$ (see Dat [12, Corollaire B.1.3]).*

Any supercuspidal irreducible representation is cuspidal. But it may happen that a cuspidal irreducible representation is not supercuspidal.

Example 4.3. *Assume that $G = \mathrm{GL}_2(F)$ and that the cardinality of the residue field of F has order 2 in $\mathbb{F}_l^\times$. Write P for the group of upper triangular matrices of G and L for the subgroup of diagonal matrices. Then the induced representation $i_{L,P}^G(1)$ of the trivial character of L is indecomposable and has length 3. Its unique irreducible subrepresentation is the trivial character of G, and its unique irreducible quotient is the unramified character of order 2 of G. The remaining irreducible subquotient is cuspidal, but not supercuspidal.*

It is thus not possible to define the cuspidal part of a smooth l-modular representation of G. The notion of cuspidal representation has to be replaced by that of supercuspidal representation. It is thus natural to ask whether or not there is such a thing as the "supercuspidal support" of an irreducible l-modular representation of G. Given a smooth irreducible representation (π, V) of a p-adic reductive group G, there is a parabolic pair (P, L) of G and a supercuspidal irreducible representation ρ of L such that π is a subquotient of $i_{L,P}^G(\rho)$. It is conjectured in [22] that the G-conjugacy class of the pair (L, ρ) is unique (and thus called the supercuspidal support of π). This conjecture is known to be true only for very few groups:

1. for $\mathrm{GL}_n(F)$ (see Vignéras [22]).
2. for inner forms of $\mathrm{GL}_n(F)$ (see Mínguez-Sécherre [17]).

Both cases require a substantial use of the l-modular theory of Bushnell-Kutzko's types. For other classical groups, one does not know whether the conjecture is true. Even for finite reductive classical groups, the conjecture is not known to be true, except for GL_n.

Finally I will say a word about the decomposition of $\mathcal{R}_{\overline{\mathbb{F}}_l}(G)$. Vignéras [21] has proved that there is a decomposition of $\mathcal{R}_{\overline{\mathbb{F}}_l}(G)$ into a product of subcategories $\mathcal{R}_{\overline{\mathbb{F}}_l}(G)_r$ indexed by rational numbers $r \geq 0$. These subcategories are made of those smooth representations of level r, and they are not indecomposable.

Even for $G = \mathrm{GL}_n(F)$ and its inner forms, one does not know whether the category $\mathcal{R}_{\overline{\mathbb{F}}_l}(G)$ decomposes into a product of indecomposable summands. The answer is expected to be yes, and these summands are expected to be made of those smooth representations whose all irreducible subquotients have a given inertial class of supercuspidal support. For $G = \mathrm{GL}_n(F)$ and its inner forms, this question is studied in a work in progress by S. Stevens and V. Sécherre.

4.2 The $l = p$ case

In this case, there is no Haar measure on G, thus there is no Hecke algebra $\mathcal{H}(G)$. But there is the following spectacular result.

Proposition 4.4. *Let (π, V) be a smooth representation of G on an $\overline{\mathbb{F}}_p$-vector space V. Assume V is nonzero. Then for all pro-p-subgroups K of G, one has $V^K \neq \{0\}$.*

We now restrict ourselves to the group $G = \mathrm{GL}_n(F)$. Thanks to Proposition 4.4, the pro-p-Iwahori subgroup I_1, made of all matrices in $\mathrm{GL}_n(O_F)$ whose reduction modulo P_F is upper triangular unipotent, plays an important role, and one can define the relative Hecke algebra $\mathcal{H} = \mathcal{H}(G, I_1)$, made of all $\overline{\mathbb{F}}_p$-valued functions on G that are I_1-biinvariant and compactly supported. The functor:

$$V \mapsto V^{I_1} \tag{4}$$

from $\mathcal{R}_{\overline{\mathbb{F}}_p}(G)$ to the category of right $\mathcal{H}$-modules gives a relationship between p-modular representations of G and $\mathcal{H}$-modules, but it is not exact in general. If (π, V) is a smooth representation of G generated by its I_1-invariant vectors, and if V^{I_1} is a simple $\mathcal{H}$-module, then (π, V) is irreducible. But there are known examples of irreducible (π, V) such that V^{I_1} is nonzero and nonsimple.

It is still possible to define parabolic functors $i^G_{L,P}$ and $r^G_{L,P}$. The induction functors are exact, but not the Jacquet functors (which are badly behaved and not much used); one can define "ordinary part" functors which occasionally play the role of Jacquet functors (see [13]).

It is not known whether or not any irreducible p-modular representation has a central character or is admissible (except for $n = 2$ and $F = \mathbb{Q}_p$, see [1]).

Finally, I will end this course by the following theorem.

Theorem 4.5 ([18, 15])**.** *Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$. The functor (4) induces an equivalence between:*

1. *the category of (smooth, p-modular) representations of G that are generated by their I_1-invariant vectors;*

2. *the category of right $\mathcal{H}$-modules.*

Remark 4.6. *The group considered in [18] is not $\mathrm{GL}_2(\mathbb{Q}_p)$, but its quotient by $p^{\mathbb{Z}}$.*

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