Versailles,

June 15, 2017

# Rigidity of the conservation laws for the Nonlinear Schrödinger equation

Nikolay Tzvetkov

Université de Cergy-Pontoise

## The linear periodic Shrödinger equation

- In dimension 1, the linear periodic Schrödinger equation reads

$$\partial_t u = i\,\partial_x^2 u,\ \ u(0,x) = u_0(x), \tag{1}$$

where $i = \sqrt{-1}$, $t \in \mathbb{R}$, $x \in \mathbb{T} = \mathbb{R}|(2\pi\mathbb{Z})$ and $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$.

- For $u_0 \in C^\infty(\mathbb{T})$ the solution of (1) is given by the exponential sum

$$u(t,x) = \sum_{n\in\mathbb{Z}} e^{-itn^2}\, e^{inx}\, \widehat{u_0}(n),$$

where $\widehat{u_0}(n)$ is the $n$'th Fourier coefficient of $u_0(x)$, i.e.

$$\widehat{u_0}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} u_0(x)\, dx.$$

- Observe that $u(t,x)$ is $2\pi$-periodic in time : $u(t+2\pi,x) = u(t,x)$.

## Conservation of the Sobolev norms

• If a function $u : \mathbb{T} \to \mathbb{C}$ has a Fourier expansion

$$u(x) = \sum_{n\in\mathbb{Z}} e^{inx}\, \widehat{u}(n)$$

then for $s \in \mathbb{R}$, the Sobolev norm $H^s$ of $u$ is defined by

$$\|u\|_{H^s}^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^s |\widehat{u}(n)|^2 \,.$$

For $s = 0$, we recover an equivalent to the $L^2$ norm and moreover

$$\|u\|_{H^1} \approx \|u\|_{L^2} + \|u'\|_{L^2}, \quad \|u\|_{H^2} \approx \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2}, \quad \text{etc.}$$

• It is now clear that the above solution of the linear periodic Schrödinger equation satisfies

$$\|u(t,\cdot)\|_{H^s} = \|u_0\|_{H^s}, \quad \forall\, t \in \mathbb{R}. \tag{2}$$

• We can therefore uniquely extend the solution map $u_0(x) \mapsto u(t,x)$ to a continuous map from $H^s(\mathbb{T})$ to $C(\mathbb{R}; H^s(\mathbb{T}))$, $s \in \mathbb{R}$. Moreover the $H^s(\mathbb{T})$ norm is preserved, i.e. we have (2).

## The Zygmund argument

• Let $u(t,x)$ be the above defined solution of the linear Schrödinger equation with $u_0 \in C^\infty(\mathbb{R})$. As we do for Gauss sums, we compute

$$u^2(t,x) = \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} e^{-it(n_1^2+n_2^2)} \, e^{i(n_1+n_2)x} \, \widehat{u_0}(n_1) \, \widehat{u_0}(n_2) \, .$$

• We now reorganise the double sum as follows :

$$u^2(t,x) = \sum_{m_1\in\mathbb{Z}} \sum_{m_2\in\mathbb{Z}} e^{-im_1 t} e^{im_2 x} \Big( \sum_{(n_1,n_2)\in A(m_1,m_2)} \widehat{u_0}(n_1) \, \widehat{u_0}(n_2) \Big),$$

where

$$A(m_1,m_2) \equiv \Big( (n_1,n_2) \in \mathbb{Z}^2 \, : \, n_1^2 + n_2^2 = m_1, n_1 + n_2 = m_2 \Big).$$

For fixed $(m_1,m_2) \in \mathbb{Z}^2$, the set $A(m_1,m_2)$ does not contain more than 2 elements. Therefore

$$\Big| \sum_{(n_1,n_2)\in A(m_1,m_2)} \widehat{u_0}(n_1) \, \widehat{u_0}(n_2) \Big|^2 \leq 2 \sum_{(n_1,n_2)\in A(m_1,m_2)} |\widehat{u_0}(n_1) \, \widehat{u_0}(n_2)|^2$$

## The Zygmund argument (sequel)

- Therefore, we arrive at the bound

$$\|u^2(t,x)\|^2_{L^2_{t,x}} \lesssim \sum_{m_1\in\mathbb{Z}} \sum_{m_2\in\mathbb{Z}} \sum_{(n_1,n_2)\in A(m_1,m_2)} |\widehat{u_0}(n_1)\,\widehat{u_0}(n_2)|^2\,.$$

- We now reorganise the sum again and we get $\|u_0\|^2_{L^2} \times \|u_0\|^2_{L^2}$. Thus

$$\|u(t,x)\|^4_{L^4_{t,x}} = \|u^2(t,x)\|^2_{L^2_{t,x}} \lesssim \|u_0\|^4_{L^2}.$$

- By a density argument, we get the following remarkable property : If $u(t,x)$ solves the linear periodic Schrödinger equation with initial data

$$u_0 \in L^2(\mathbb{T})$$

then

$$\|u(t,\cdot)\|_{L^4} < \infty, \quad \text{a.s in } t \in \mathbb{R}.$$

This property can only hold only a.s. because of the strict inclusion

$$L^4(\mathbb{T}) \subset L^2(\mathbb{T}).$$

## The linear Shrödinger equation on the line

• Consider now the $1d$ linear Schrödinger equation on the real line

$$\partial_t u = i\,\partial_x^2 u,\ u(0,x) = u_0(x) \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{3}$$

If $u_0$ is in the Schwartz class then the solution is given by

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi^2}\, e^{ix\xi}\,\widehat{u_0}(\xi)d\xi,$$

where $\widehat{u_0}(\xi)$, $\xi \in \mathbb{R}$ is the Fourier transform of $u_0$, defined by

$$\widehat{u_0}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}u_0(x)dx\,.$$

• The Sobolev norm $H^s$ of functions on $\mathbb{R}$ is now defined by

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}}(1+\xi^2)^s|\widehat{f}(\xi)|^2 d\xi\,.$$

Since

$$\widehat{u(t,\cdot)}(\xi) = e^{-it\xi^2}\,\widehat{u_0}(\xi) \Longrightarrow |\widehat{u(t,\cdot)}(\xi)| = |\widehat{u_0}(\xi)|$$

the solution of (3) satisfies

$$\|u(t,\cdot)\|_{H^s} = \|u_0\|_{H^s}\,.$$

## The dispersion

- By applying a stationary phase estimate to

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi^2} e^{ix\xi}\,\widehat{u_0}(\xi)d\xi,$$

we obtain that there is $c \in \mathbb{C}$ and $C > 0$ such that for such that for every $t \geq 1$, every $x \in \mathbb{R}$,

$$\left| u(t,x) - c\frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}\,\widehat{u_0}(x/2t)\right| \leq Ct^{-\frac{3}{4}}\|xu_0\|_{L^2}\,.$$

In particular, for $t \geq 1$,

$$|u(t,x)| \leq C(u_0)\,t^{-\frac{1}{2}}\,.$$

- Therefore the solution disperses keeping the $H^s$ norms conserved.
- Another manifestation of the dispersion is the Strichartz estimate

$$\|u(t,x)\|_{L^6(\mathbb{R}\times\mathbb{R})} \leq C\|u_0(x)\|_{L^2(\mathbb{R})}\,.$$

## A fully non linear model

•Consider the equation

$$\partial_t u = -i|u|^2 u, \quad u(0,x) = u_0(x).$$

• For $u_0 \in L^2$, the solution is given by :

$$u(t,x) = e^{-it|u_0(x)|^2} u_0(x)\,.$$

• Then

$$\partial_x u(t,x) = e^{-it|u_0(x)|^2} \Big(\partial_x u_0(x) - itu_0(x)\,\partial_x(|u_0(x)|^2)\Big).$$

• Therefore for $u_0(x)$ such that $|u_0(x)|$ is not a constant, there exists $C > 0$ and $A \geq 1$ such that for $t \geq A$,

$$\|u(t,\cdot)\|_{H^1} \geq Ct,$$

i.e. the $H^1$ norm grows in time ! Similarly for $H^s$, $s \geq 0$ initial data

$$\|u(t,\cdot)\|_{H^s} \geq Ct^s.$$

• A remarkable work by P. Gérard- S. Grellier shows that the growth may become even faster (exponential ?) if one "truncates" $|u|^2 u$ ...

## The 1d Nonlinear Schrödinger equation (NLS)

• We considered so far the linear model

$$\partial_t u = i\,\partial_x^2 u$$

and the fully nonlinear model

$$\partial_t u = -i|u|^2 u$$

• The $1d$ NLS is obtained when one takes into account both effects :

$$\partial_t u = i\,\partial_x^2 u - i|u|^2 u$$

or equvalently

$$i\partial_t u + \partial_x^2 u = |u|^2 u\,.$$

• For the linear model the Sobolev noms $H^s$ of the solutions remain bounded while for the fully nonlinear model they grow as far as $s > 0$.
• The question we discuss today is which effect dominates in the context of NLS.

## Global well-posedness and basic conservation laws for NLS

• Thanks to the $1d$ Sobolev emebedding $H^s \subset L^\infty$, $s > 1/2$ , we can easily solve locally in $H^s$, $s > 1/2$ the problem

$$i\partial_t u + \partial_x^2 u = |u|^2 u\,. \tag{4}$$

• Multiply (4) with $\bar{u}$, $i\partial_t \bar{u}$, integrate over $x$ and take the imaginary part. It comes :

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = 0, \quad \frac{d}{dt}\Big(\|\partial_x u(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|u(t,\cdot)\|_{L^4}^4\Big) = 0\,.$$

• One can deduce the second conservation law as the Hamiltonian conservation resulting from the Hamiltonian formulation of NLS.

• Therefore, for $s \geq 1$ we can extend globally in time the local solutions. Moreover, the $L^2$ and the $H^1$ norms of the solutions remain bounded in time. Therefore, concerning the $H^1$ norm, the linear effect dominates.

**Question :** What about the $H^s$, $s > 1$ norms ?

**Remark :** The question of growing Sobolev norms may be seen as a competition between the kinetic and the potential energies.

## Higher order conservation laws for 1d NLS

• Using the Lax representation of the 1d NLS, Zakharov-Shabat (1972) obtained that if $u$ is an $H^s$, $s \geq 2$ solution of

$$i\partial_t u + \partial_x^2 u = |u|^2 u$$

then

$$\frac{d}{dt}\Big( \|\partial_x^2 u\|_{L^2}^2 + 2\|\mathrm{Re}(\partial_x u\,\bar{u})\|_{L^2}^2 + 3\|u\partial_x u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^6}^6 \Big) = 0\,.$$

Here $x$ can be both in $\mathbb{T}$ or $\mathbb{R}$.

• Therefore the $H^2$ norms of the solutions remain bounded in time.

• Similarly one gets uniform in time bounds for the $H^s$ norms, $s = 3, 4, 5, \ldots$.

• Recent work (2016) by Koch-Tataru extends these bounds for all $s \geq 0$ in the case $x \in \mathbb{R}$ (for $x \in \mathbb{T}$, there is an earlier work by Grebert-Kappeler).

## Conclusion of the 1d analysis

• In summary, for the $1d$ NLS both on $\mathbb{R}$ and $\mathbb{T}$, the linear effect dominates concerning the bounds on the Sobolev norms of the solutions.

• What happens in higher dimensions, i.e. for the equation

$$i\partial_t u + \Delta u = |u|^2 u,$$

where $\Delta$ is the Laplace operator ?

• **Remark.** In higher dimensions, even the global well-posedness is a quite nontrivial problem.

# The $3d$ NLS

• Let $(M, g)$ be a smooth $3d$ riemannian manifold with a Laplace-Beltrami operator $\Delta$ . Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times M \to \mathbb{C}. \tag{5}$$

• As in $1d$, in the context of (5), we again have the conserved quantities

$$\|U\|_{L^2(M)}, \quad \|U\|^2_{H^1(M)} + \frac{1}{2}\|U\|^4_{L^4(M)}.$$

**Theorem 1 (Burq-Gérard-Tz. 2001)**

*Suppose that $M$ is compact without boundary. For $s \geq 1$ and $U_0 \in H^s(M)$ there is a unique global solution of (5) in $C(\mathbb{R}; H^s(M))$. The dependence with respect to the initial data in continuous. The $L^2$ and the $H^1$ norms of the solutions are uniformly bounded in time.*

• The result remains true for non compact manifolds with a controlled behaviour at infinity such as $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{T}^2$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times S^2$ or a long range perturbation of $\mathbb{R}^3$ outside a compact set.

**Question :** Do the $H^s$ norms, $s \neq 0, 1$ remain bounded ?

## The $3d$ NLS on $\mathbb{R}^3$

Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}. \tag{6}$$

**Theorem 2 (Ginibre-Velo, Bourgain, Dodson)**
*For $s > 5/7$ the problem (6) is globally well-posed in $H^s(\mathbb{R}^3)$. Moreover, for every $U_0 \in H^s$ there is $C > 0$ such that for every $t \in \mathbb{R}$ the solution of (6) satisfies*

$$\|U(t,\cdot)\|_{H^s(\mathbb{R}^3)} \leq C. \tag{7}$$

For $s \geq 1$, one may proceed in two steps :
1) Using Morawetz identities (a way of exploiting the good sign of the nonlinearity in a dispersive estimate) one first shows that the $L^p(\mathbb{R}^3)$, $p \in (2,6)$ norms of the solution go to zero as $t$ tends to infinity.
2) Then by a perturbative analysis one reinforces this information to a control on space-time norms like $L^{10}(\mathbb{R} \times \mathbb{R}^3)$ of the solutions which in turn implies (7).

## The 3$d$ NLS on $\mathbb{R}\times\mathbb{T}^2$

• Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R}\times(\mathbb{R}\times\mathbb{T}^2) \to \mathbb{C}. \qquad (8)$$

1) The problem (8) is locally well-posed in $H^s(\mathbb{R}\times\mathbb{T}^2)$, $s > 1/2$ (ideas by Bourgain)
2) It is ill-posed for $s \in (0, 1/2)$ (ideas by Lebeau).
3) It is globally-well-posed for $s > 5/6$ (ideas by Tao et al.).

**Theorem 3 (Pausader-Tz. 2017)**
*For every $s \in (1/2, \infty)$, $s \neq 1$ there exists $U_0 \in H^s(\mathbb{R}\times\mathbb{T}^2)$ such that the corresponding solution of (8) is globally defined and*

$$\limsup_{t\to\infty} \|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)} = +\infty\,.$$

• Recall that for $s \geq 1$, the conservation laws provide an a priori bound on the $H^1(\mathbb{R}\times\mathbb{T}^2)$ norm. We also always have an a priori bound on the $L^2(\mathbb{R}\times\mathbb{T}^2)$ norm. A nonlinear interpolation is therefore impossible.
• Previous work by Hani-Pausader-Tz.-Visciglia 2013, obtained this result for $s \geq 30$.

## Reduction of the problem

• Let $U(t)$ be a solution of the cubic defocusing NLS, posed on $\mathbb{R}\times\mathbb{T}^2$. Then $F(t) = e^{-it\Delta}U(t)$ solves

$$i\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)],$$

where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F,G,H] := e^{-it\Delta}\Big(e^{it\Delta}F \cdot e^{-it\Delta}\overline{G} \cdot e^{it\Delta}H\Big).$$

• Denote by $\widehat{F}_p(\xi)$ or $\mathcal{F}(F)(\xi,p)$ the Fourier transform on $\mathbb{R}\times\mathbb{T}^2$ of $F$. Then one computes :

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-p_1+p_2-p_3=0} e^{it\left[|p|^2-|p_1|^2+|p_2|^2-|p_3|^2\right]} \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{F}_{p_1}(\xi-\eta)\overline{\widehat{G}_{p_2}}(\xi-\eta-\kappa)\widehat{H}_{p_3}(\xi-\kappa)d\kappa d\eta\,.$$

## Reduction of the problem (sequel)

• Ignoring the time oscillations (normal form reduction) and a stationary phase argument ($t \gg 1$) suggests to define $\mathcal{R}$ as

$$\mathcal{F}\mathcal{R}[F,G,H](\xi,p) := \sum_{\substack{p+p_2=p_1+p_3 \\ |p|^2+|p_2|^2=|p_1|^2+|p_3|^2}} \widehat{F}_{p_1}(\xi)\overline{\widehat{G}_{p_2}}(\xi)\widehat{H}_{p_3}(\xi)$$

and one expects that the nonlinearity can be decomposed as follows

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t}\mathcal{R}[F,G,H] + \text{ better terms}$$

• We therefore define the resonant system as

$$i\partial_t G(t) = \mathcal{R}[G(t),G(t),G(t)]\,.$$

• The dependence on $\xi$ is merely parametric.

• We prove that given a solution $G$ of the resonant system, bounded in "some norm", there is a solution of the true problem "close" to $G(\pi \ln(t))$ for $t \gg 1$.

## How we justify the normal form reduction and the stationary phase ?

• This is a long argument, using the following tools :

1) Variants of the Zygmund argument, we saw in the beginning of the lecture

2)Variants of the dispersive estimate, we saw in the beginning of the lecture

3) A new lemma of Christ-Kiselev type

4) Almost orthogonality arguments, inspired by the work of Bourgain

5) The Bourgain/Tataru spaces

• A combination if 1) and 2) provides a low regularity in the periodic variable $L^4_{t,x,y}$ dispersive estimate which is very efficient.

## Reduction to the resonant system on $\mathbb{T}^2$

• We take initial data of the resonance system of the form

$$G_0(x,y) = \mathcal{F}_{\mathbb{R}}^{-1}(\varphi)(x)g(y), \quad x \in \mathbb{R}, y \in \mathbb{T}^2,$$

with $\varphi$ real valued. The solution $G(t)$ to the resonance system with initial data $G_0(x,y)$ as above is given in Fourier space by

$$\widehat{G}_p(t,\xi) = \varphi(\xi)a_p(\varphi(\xi)^2 t), \quad a_p(0) = \mathcal{F}_{\mathbb{T}^d}(g)(p),$$

where the vector $(a_p)_{p\in\mathbb{Z}^2}$ solves the *resonant equation*

$$i\partial_t a_p(t) = \sum_{\substack{p+p_2=p_1+p_3 \\ |p|^2+|p_2|^2=|p_1|^2+|p_3|^2}} a_{p_1}(t)\overline{a_{p_2}(t)}a_{p_3}(t)\,.$$

• In particular, if $\varphi = 1$ on an open interval $I$, then $\widehat{G}_p(t,\xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. We can therefore apply the following result.

**Theorem 4 (growth for the resonant equation)**
*Let $s > 0$, $s \neq 1$. There exist global solutions to the resonant equation in $C(\mathbb{R}; h^s)$ such that*

$$\sup_{t \geq 0} \|a_p(t)\|_{h^s} = \infty$$

*but for every $\varepsilon > 0$*

$$\sup_{t \geq 0} \|a_p(t)\|_{h^{s-\varepsilon}} < \infty.$$

- **Notation** :

$$\|a_p\|_{h^s}^2 := \sum_{p \in \mathbb{Z}^2} (1 + |p|^2)^s |a_p|^2.$$

- **Remark.** Unfortunately, we have that, $a_p(t) \notin h^\sigma$ for $\sigma > s$.

## On the analysis of the resonant equation

• The analysis of the resonant equation is inspired by a work of Colliander-Keel-Staffilani-Takaoka-Tao. Two important aspects are:

1) There are many invariant subspaces for the resonant equation.

2) There is a superposition principle : for some initial data it "behaves as a linear equation".