Multidimensional Borg-Levinson type theorems

Mourad Choulli

Institut Élie Cartan de Lorraine

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Summary

1 The classical Borg-Levinson theorem

2 Nachman-Sylvester-Uhlmann's result

3 Isozaki's idea

4 A stability result by Alessandrini and Sylvester

5 Extensions by M. C. and P. Stefanov

6 Kavian-Kian-Soccorsi's idea

7 Extension to a magnetic Schödinger operator on compact Riemannian manifold

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. G. Borg : *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta. Math. 78 (1946), 1-96.

. N. Levinson : *The inverse Sturm-Liouville problem*, Mat. Tidsskr. B. (1949), 25-30.

Let $q \in L^\infty((0,1))$ and $y(x,\lambda)$ be the solution of the IVP

$$\begin{cases} -y'' + qy = \lambda y \ \text{ in } (0,1), \\ y(0,\lambda) = 0, \ y'(0,\lambda) = 1. \end{cases}$$

Define the sequence $(\lambda_k(q))_{k\geq 1}$ of Dirichlet eigenvalues by

$$y(1, \lambda_k(q)) = 0$$

and the norming constants $c_k(q)$, $k \geq 1$, by

$$c_k(q) = \int_0^1 y^2(x, \lambda_k(q))dx.$$

The classical Borg-Levinson theorem is

Theorem 1

If $q_1,\ q_2 \in L^\infty((0,1))$ are such that

$$\lambda_k(q_1) = \lambda_k(q_2) \ \text{ and } \ c_k(q_1) = c_k(q_2), \quad k \geq 1,$$

then $q_1 = q_2$.

We paraphrase Theorem 1 by

Corollary 1

Let $q_1,\ q_2 \in L^\infty((0,1))$ satisfying

$$\lambda_k(q_1) = \lambda_k(q_2) \ \text{ and } \ y'(1, \lambda_k(q_1)) = y'(1, \lambda_k(q_2)), \quad k \geq 1.$$

Then $q_1 = q_2$.

This corollary has a direct generalization to higher dimensions.

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A. Nachmann, J. Sylvester and G. Uhlmann : *An n-dimensional Borg-Levinson theorem*, Commun. Math. Phys. 115 (1988), 595-605.

Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with boundary Γ.

To $q \in L^\infty(\Omega)$, we associate the unbounded operator

$$A(q) = -\Delta + q, \quad D(A(q)) = H_0^1(\Omega) \cap H^2(\Omega).$$

The spectrum of $A(q)$ consists in a sequence of eigenvalues $\lambda(q) = (\lambda_k(q))$, counted according to their multiplicity :

$$-\infty < \lambda_1(q) \leq \lambda_2(q) \leq \ldots \leq \lambda_k(q) \to +\infty.$$

An orthonormal basis of eigenfunctions will denoted by $\varphi(q) = (\varphi_k(q))$. By the classical H^2-regularity theorem, $\varphi_k(q) \in H^2(\Omega)$ and therefore $\partial_\nu \varphi_k(q) \in H^{\frac{1}{2}}(\Gamma)$. We set $\partial_\nu \varphi(q) = (\partial_\nu \varphi_k(q))$.

Theorem 2

Let q_1, $q_2 \in L^\infty(\Omega)$ and $\varphi(q_1)$ an orthonormal basis for q_1. Assume that $\lambda(q_1) = \lambda(q_2)$ and there exists an orthonormal basis $\varphi(q_2)$ such that $\partial_\nu \varphi(q_2) = \partial_\nu \varphi(q_1)$. Then $q_1 = q_2$.

The main idea in the proof of Theorem 2 : a formula providing the relationship between the spectral data and the family of spectral DN maps.

For each $f \in H^{1/2}(\Gamma)$ and $\lambda \in \rho(A(q))$, the BVP

$$\begin{cases} (-\Delta + q - \lambda)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma \end{cases}$$

has a unique solution $u(q,\lambda)(f) \in H^1(\Omega)$ and

$$\Lambda(q,\lambda) : f \to \partial_\nu u(q,\lambda)(f)$$

defines a bounded operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.

Lemma 1

Let $q \in L^\infty(\Omega)$, $\varphi(q)$ *an orthonormal basis,* $f \in H^{1/2}(\Gamma)$, $m > \frac{n}{2} + \frac{3}{4}$ *and* $\lambda \in \rho(A(q))$. *Then*

$$\Lambda^{(m)}(q,\lambda)f = -m! \sum_{k \geq 1} \frac{1}{(\lambda_k(q) - \lambda)^{m+1}} \left(\int_\Gamma f \partial_\nu \varphi_k(q) \right) \partial_\nu \varphi_k(q),$$

where the series converges absolutely in $L^2(\Gamma)$.

Lemma 2

Let q_1, $q_2 \in L^\infty(\Omega)$ with $\|q_1\|_\infty + \|q_2\|_\infty \leq c$. For any positive integer ℓ and $0 < \epsilon < 1/2$, there exists a constant $C_\epsilon > 0$, that can depend only on c, Ω, ℓ and ϵ, so that

$$\|\Lambda^{(j)}(q_1,\lambda) - \Lambda^{(j)}(q_2,\lambda)\|_{\mathscr{B}(H^{1/2}(\Gamma),L^2(\Gamma))} \leq \frac{C_\epsilon}{|\Re\lambda|^{j+\sigma_\epsilon}},$$

$$0 \leq j \leq \ell,\ \Re\lambda \leq -2c,\ \sigma_\epsilon = \frac{1-2\epsilon}{4}.$$

- Lemma 1 $\Longrightarrow \lambda \to \Lambda(q_1,\lambda) - \Lambda(q_2,\lambda)$ is a polynomial function.
- Lemma 2 $\Longrightarrow \Lambda(q_1,\lambda) = \Lambda(q_2,\lambda)$.
- A classical argument based on CGO solutions gives

$$\Lambda(q_1,\lambda) = \Lambda(q_2,\lambda),\ -\lambda \gg 1 \Longrightarrow q_1 = q_2.$$

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. H. Isozaki : *some remaks on the multi-dimensional Borg-Levinson theorem* J. Math. Kyoto Univ. 31 (3) (1991), 743-753.

For $N \geq 1$, we set

$$\lambda^N(q) = (\lambda_k(q))_{k\geq N}, \quad \partial_\nu \varphi^N(q) = (\partial_\nu \varphi_k(q))_{k\geq N}.$$

Theorem 3

Fix $N \geq 1$. Let q_1, $q_2 \in L^\infty(\Omega)$ and $\varphi(q_1)$ an orthonormal basis for q_1. Assume that $\lambda^N(q_1) = \lambda^N(q_2)$ and there exists $\varphi(q_2)$ an orthonormal basis for q_2 such that $\partial_\nu \varphi^N(q_2) = \partial_\nu \varphi^N(q_2)$. Then $q_1 = q_2$.

Let $e_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\omega\cdot x}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\omega \in \mathbb{S}^{n-1}$. Consider

$$S(q)(\lambda,\omega,\theta) = \int_\Gamma \Lambda(q,\lambda)(e_{\lambda,\omega})e_{\lambda,-\theta},$$
$$\lambda \in \rho(A(q)) \setminus (-\infty, 0], \ \omega, \ \theta \in \mathbb{S}^{n-1}.$$

Key formula in the proof of Theorem 3 :

$$\begin{aligned} S(q)(\lambda,\omega,\theta) &= -\frac{\lambda}{2}|\theta-\omega|^2\int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x} \\ &+ \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x}q(x) - \int_\Omega R(q,\lambda)(qe_{\lambda,\omega})qe_{\lambda,-\theta}, \end{aligned}$$

where $R(q,\lambda)=(A(q)-\lambda)^{-1}$.

Born approximation : Fix $\xi\in\mathbb{S}^{n-1}$ and $\eta\in\mathbb{S}^{n-1}$, $\eta\perp\xi$. For $\tau>1$, let

$$\theta_\tau = c_\tau\eta+\frac{1}{2\tau}\xi,\quad \omega_\tau=c_\tau\eta-\frac{1}{2\tau}\xi,\quad \sqrt{\lambda_\tau}=\tau+i,$$

where $c_\tau=\sqrt{1-\frac{1}{4\tau^2}}$.

Then

$$\lim_{\tau\to+\infty}S(q)(\lambda_\tau,\omega_\tau,\theta_\tau) = -\frac{|\xi|^2}{2}\int_\Omega e^{-ix\cdot\xi}+\int_\Omega e^{-ix\cdot\xi}q(x).$$

A generalization of Theorems 2 and 3 to unbounded potentials was recently established in

- V. Pohjola *Multidimensional Borg-Levinson theorem for unbounded potentials*, arXiv : 1612.02937.

Precisely, Theorem 2 still holds if $n \geq 3$ and the potentials are in $L^{\frac{n}{2}}(\Omega)$; while Theorem 3 is valid for potentials in $L^p(\Omega)$ with $p = \frac{n}{2}$ if $n \geq 4$ and $p > \frac{n}{2}$ if $n = 3$.

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. G. Alessandrini and J. Sylvester *Stability for a multidimensional inverse spectral theorem*, Commun. PDE, 15 (5) (1990), 711-736.

In the sequel we always need to assume that 0 is not an eigenvalue of $A(q)$, $q = q_1$ or q_2. This is not really a restriction. In fact, since $q_1 - q_2 = (q_1 + \mu) - (q_2 + \mu)$, we can choose μ such that 0 is not an eigenvalue of $A(q + \mu)$, $q = q_1$ or q_2.

Theorem 4

Let $q_1, q_2 \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$ with $\|q_1\|_\alpha + \|q_2\|_\alpha \leq c$. Then, there exist positive constants A, B, C and $0 < \delta < 1$ such that, for every $N > 0$,

$$\|q_1 - q_2\|_\infty \leq C\left(N^A \epsilon^\delta + N^{-B}\right),$$

where

$$\epsilon = \sup_{k \leq N} |\lambda_k(q_1) - \lambda_k(q_2)| + \sup_{k \leq N} \|\partial_\nu \varphi_k(q_1) - \partial_\nu \varphi_k(q_2)\|_\infty.$$

Theorem 4 was reformulated in

. M. C. *Une introduction aux problèmes inverses elliptiques et paraboliques*, SMAI-Springer-Verlag, Berlin, 2009.

by introducing appropriate metrics for the sequence of eigenvalues and the sequence of the normal derivative of eigenfunctions. Fix $\frac{n}{2}+1<\zeta\leq n+1$ and set

$$d_1(\lambda(q_1),\lambda(q_2)) = \|\lambda(q_1)-\lambda(q_2)\|_\infty,$$

$$d_2(\partial_\nu\varphi(q_1),\partial_\nu\varphi(q_2)) = \sum_{k\geq 1} k^{-\frac{2\zeta}{n}} \|\varphi_k(q_1)-\varphi_k(q_2)\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Theorem 5

Let $q_1,\ q_2\in C^\alpha(\overline{\Omega})$, $0<\alpha<1$ *and* $\|q_1\|_\alpha+\|q_2\|_\alpha\leq c$. *For any* $0\leq\theta<\frac{1}{2}$, *there exists* $C>0$ *such that*

$$\|q_1-q_2\|_\infty\leq C\left[d_1(\lambda(q_1),\lambda(q_2))+d_2(\partial_\nu\varphi(q_1),\partial_\nu\varphi(q_2))\right]^\beta,$$

where $\beta=\left(1-\frac{4}{(1-2\theta)+n+4}\right)\frac{\alpha\min(2\alpha,1)}{(2\alpha+n)(2n+5)(n+\alpha+\frac{15}{2})}$.

We have a version of this theorem when the Laplacian is changed to the Laplace-Beltrami operator defined on a simple Riemannian manifold :

. M. Bellassoued and D. Dos Santos Ferreira, *Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map*, IPI 5 (4) (2011), 745-773.

. The original idea by Alessandrini and Sylvester for proving their stability estimate is based on the relationship between the spectral data and the DN map for the wave equation.

Fix $T > 0$. Set $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$ and consider the IBVP for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta + q)u = 0 \ \text{ in } Q, \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, \\ u_{|\Sigma} = f. \end{cases} \tag{1}$$

Let

$$\Xi = \{h \in H^1(0,T;H^{\frac{3}{2}}(\Gamma)) \cap H^2(0,T;L^2(\Gamma));\ h(\cdot,0) = \partial_t h(\cdot,0) = 0\}.$$

For $f \in \Xi$, the IBVP (2) has a unique solution

$$u(q,f) \in L^2(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega))$$

and the hyperbolic DN operator

$$H(q) : f \in \Xi \to \partial_\nu u(q,f) \in L^2(0,T;H^{\frac{1}{2}}(\Gamma))$$

is bounded.

Set

$$\Xi_0 = \{h \in H^{2n+4}(0,T;H^{\frac{3}{2}}(\Gamma));\ h(\cdot,0) = \partial_t^j h(\cdot,0) = 0,\ 0 \le j \le 2n+3\}.$$

Using CGO solutions and properties of the X-ray tranform, we prove (with T large enough)

$$\|q_1 - q_2\|_\infty \le C\|H(q_1) - H(q_2)\|_\theta^\kappa,$$

where

$$\kappa = \frac{\alpha \min(2\alpha, 1)}{(2\alpha + n)(2n+5)(n + \alpha + \frac{15}{2})}$$

and $\|\cdot\|_\theta$ is the operator norm between Ξ_0 and $L^2(0, T; H^\theta(\Gamma))$.

. From the wave equation to the spectral problem :

$$H(q)f = \sum_{j=0}^{n+1} \left[\frac{d^j}{d\lambda^j} \Lambda(q, \lambda) \right]_{|\lambda=0} (-\partial_t^2 f) + \mathcal{R}(q)f$$

with

$$\mathcal{R}(q)f = \sum_{k\geq 1} \frac{1}{\lambda_k(q)^{n+\frac{5}{2}}} \partial_\nu \varphi_k(q)$$

$$\times \int_0^t \langle -\partial_s^{2n+4} f(\cdot, s), \partial_\nu \varphi_k(q) \rangle \sin \sqrt{\lambda_k(q)} (t-s) ds.$$

The problem becomes more difficult if we replace $\partial_\nu \varphi_k(q)$ by $\partial_\nu \varphi_k(q)_{|\Gamma_0}$, where Γ_0 is an open subset of Γ.

We need to quantify the unique continuation for the wave equation from Cauchy data on the sub-boundary Γ_0.

This is done as follows : we transform the wave equation to an elliptic equation by means of a FBI transform. The quantification of the unique continuation, for the elliptic equation, from the Cauchy data on Γ_0 is obtained by a classical method based on a Carleman estimate.

Theorem 6

Let ω be a neighborhood of Γ in $\overline{\Omega}$ and $s > \frac{n}{2}$. There exist $C > 0$ and $\mu \in (0, 1)$ such that, for any q_1, $q_2 \in H^s(\Omega)$ satisfying $q_1 = q_2$ in ω, we have

$$\|q_1 - q_2\|_\infty \le |\ln |\ln d||^\mu ,$$

where

$$d = \|\lambda(q_1) - \lambda(q_2)\|_\infty + \sum_{k\ge 1} k^{-\frac{2\zeta}{n}} \|\partial_\nu \varphi_k(q_1)_{|\Gamma_0} - \partial_\nu \varphi_k(q_2)_{|\Gamma_0}\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

Theorem 6 was proved in

- M. Bellassoued, M. C. and M. Yamamoto *Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem*, J. Diff. Equat. 247 (2) (2009) 465-494.

Under an additional condition in terms of the X-ray transform of $q_1 - q_2$, we can improve the log-log type stability estimate to a log type stability estimate :

- M. Bellassoued, M. C. and M. Yamamoto *Stability estimate for a multidimensional inverse spectral problem with partial data*, J. Math. Anal. Appl. 378 (1) (2011) 184-197.

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- M. C. and P. Stefanov *Stability for the multi-dimensional Borg-Levinson theorem with partial spectral data*, Commun. PDE 38 (3) (2013) 455-476.

Theorem 7

Let q_1 , $q_2 \in L^\infty(\Omega)$. *Let, for some* $A > 0$, *and all* $k = 1, 2 \ldots,$

$$|\lambda_k(q_1) - \lambda_k(q_2)| \leq Ak^{-\alpha}, \quad \alpha > 1,$$

$$\|\partial_\nu \varphi_k(q_1) - \partial_\nu \varphi_k(q_2))\|_{L^2(\Gamma)} \leq Ak^{-\beta}, \quad \beta > 1 - \frac{1}{2n}.$$

Then $q_1 = q_2$.

Theorem 8

Fix $N \geq 1$, $c > 0$ *and* $m > \frac{n}{2} + \frac{3}{4}$. *Let* $q_1, q_2 \in L^\infty(\Omega)$ *such that* $q_1 - q_2 \in H_0^1(\Omega)$ *and*

$$\|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)} + \|q_1 - q_2\|_{H_0^1(\Omega)} \leq c.$$

Then there exist $C > 0$ *and* $0 < \gamma = \gamma(n) < 1$ *such that*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\delta^\gamma,$$

where

$$\delta = \sup_{k \geq N} |\lambda_k(q_1) - \lambda_k(q_2)| + \sum_{k \geq N} k^{-\frac{2m}{n}} \|\partial_\nu \varphi_k(q_1) - \partial_\nu \varphi_k(q_2)\|_{L^2(\Gamma)}.$$

Sketch of the proof.

- First, from the formula

$$S(q)(\lambda,\omega,\theta) = -\frac{\lambda}{2}|\theta-\omega|^2\int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x} + \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)\cdot x}q(x) - \int_\Omega R(q,\lambda)(qe_{\lambda,\omega})qe_{\lambda,-\theta},$$

and the classical estimate for the resolvent

$$\|R(q,z)\|_{\mathscr{B}(L^2(\Omega))} \leq \frac{1}{|\Im z|},\ z\notin\mathbb{R},$$

we obtain, where $\lambda = (\tau+i)^2$,

$$\left|(\widehat{q}_1-\widehat{q}_2)\left(\xi+\frac{i}{\tau}\xi\right)\right| \leq \frac{C}{\tau} + |S(q_1)(\lambda,\theta,\omega) - S(q_2)(\lambda,\theta,\omega)|.$$

- After some technical calculations, we get

$$C\|q\|^2_{L^2(\Omega)} \leq \frac{1}{\tau^{\frac{2}{n+2}}} + \tau^{\frac{n}{n+2}} |S(q_1)(\lambda,\theta,\omega) - S(q_2)(\lambda,\theta,\omega)|^2. \qquad (2)$$

- On the other hand,

$$|S(q_1)(\lambda,\theta,\omega) - S(q_2)(\lambda,\theta,\omega)| = \left| \int_\Gamma e_{\lambda,-\theta}[\Lambda(q_1,\lambda) - \Lambda(q_2,\lambda)]e_{\lambda,\omega} \right|$$
$$\leq C_\Omega \|\Lambda(q_1,\lambda) - \Lambda(q_2,\lambda)\| \, \|e_{\lambda,\omega}\|_{H^{1/2}(\Gamma)} \|e_{\lambda,-\theta}\|_{L^2(\Gamma)}.$$

- This and the following estimates

$$\|e_{\lambda,\omega}\|_{H^{1/2}(\Gamma)} \leq C\tau^{\frac{1}{2}}, \quad \|e_{\lambda,-\theta}\|_{L^2(\Gamma)} \leq C$$

imply

$$C\|q\|^2_{L^2(\Omega)} \leq \frac{1}{\tau^{\frac{2}{n+2}}} + \tau^{\frac{n}{n+2}+1} \|\Lambda(q_1,\lambda) - \Lambda(q_2,\lambda)\|^2.$$

The next step consists in estimating $\|\Lambda(q_1,\lambda)-\Lambda(q_2,\lambda)\|$ in terms of the spectral data.

- We decompose $\Lambda(q,\lambda)$, $q=q_1$ or q_2, in the following form

$$\Lambda(q,\lambda)=\widetilde{\Lambda}(q,\lambda)+\widehat{\Lambda}(q,\lambda),$$

where, for $f\in H^{1/2}(\Gamma)$,

$$\widetilde{\Lambda}(q,\lambda)f=\partial_\nu\left(\sum_{k>N}\frac{1}{\lambda_k(q)-\lambda}\left(\int_\Gamma f\partial_\nu\varphi_k(q)\right)\varphi_k(q)\right),$$

$$\widehat{\Lambda}(q,\lambda)f=\sum_{k\leq N}\frac{1}{\lambda_k(q)-\lambda}\left(\int_\Gamma f\partial_\nu\varphi_k(q)\right)\partial_\nu\varphi_k(q).$$

- Since $\|\widehat{\Lambda}(q,\lambda)-\widehat{\Lambda}(q,\lambda)\|$ is of order τ^{-1}, we can replace the last estimate by

$$C\|q\|^2_{L^2(\Omega)} \leq \frac{1}{\tau^{\frac{2}{n+2}}} + \tau^{\frac{n}{n+2}+1}\|\widetilde{\Lambda}(q_1,\lambda)-\widetilde{\Lambda}(q_2,\lambda)\|^2.$$

- For $\rho \geq 2\Re\lambda$, we set $\widetilde{\lambda} = -\rho + \lambda$. From Taylor's formula, we have, for $q = q_1$ or q_2,

$$\widetilde{\Lambda}(q,\lambda) = \mathcal{P}(q,\lambda) + \mathcal{R}(q,\lambda),$$

where

$$\mathcal{P}(q,\lambda) = \sum_{k=0}^{m-1} \frac{(\lambda-\widetilde{\lambda})^k}{k!}\widetilde{\Lambda}^{(k)}(q,\widetilde{\lambda})$$

$$\mathcal{R}(q,\lambda) = \int_0^1 \frac{(1-s)^m(\lambda-\widetilde{\lambda})^m}{(m-1)!}\widetilde{\Lambda}^{(m)}(q,\widetilde{\lambda}+s(\lambda-\widetilde{\lambda}))ds.$$

- The term $\|\mathcal{P}(q_1,\lambda)-\mathcal{P}(q_2,\lambda)\|$ can be easily estimated :

$$\|\mathcal{P}(q_1,\lambda)-\mathcal{P}(q_2,\lambda)\leq \frac{C}{\rho^{\sigma}}.$$

- By Lemma 1, we know, for $z\notin \rho(A_q)$,

$$\widetilde{\Lambda}^{(m)}(q,z)f=\sum_{k>N}\frac{1}{(\lambda_k(q)-z)^{m+1}}\left(\int_{\Gamma} f\partial_\nu\varphi_k(q)\right)\partial_\nu\varphi_k(q).$$

- Let $\mu=\mu(s)=\widetilde{\lambda}+s(\lambda-\widetilde{\lambda})=\lambda-(1-s)\rho$ and

$$N(\lambda)=\min\{k\geq N;\ \lambda_{k+1}(q)\geq 2\Re\lambda\}.$$

When $\Re\lambda\gg 1$, we decompose $\widetilde{\Lambda}^{(m)}(q,\mu)f$ as follows

$$\widetilde{\Lambda}^{(m)}(q,\mu)f=\Lambda_1^{(m)}(q,\mu)f+\Lambda_2^{(m)}(q,\mu)f,$$

where

$$\widetilde{\Lambda}_1^{(m)}(q,\mu)f = \sum_{k=N+1}^{N(\lambda)} \frac{1}{(\lambda_k(q)-\mu)^{m+1}} \left(\int_\Gamma f\partial_\nu\varphi_k(q)\right)\partial_\nu\varphi_k(q),$$

$$\widetilde{\Lambda}_2^{(m)}(q,\mu)f = \sum_{k>N(\lambda)} \frac{1}{(\lambda_k(q)-\mu)^{m+1}} \left(\int_\Gamma f\partial_\nu\varphi_k(q)\right)\partial_\nu\varphi_k(q).$$

. We have

$$\widetilde{\Lambda}_1^{(m)}(q_1,\mu)f - \widetilde{\Lambda}_1^{(m)}(q_2,\mu)f = I_1 + I_2 + I_3,$$

with

$$I_1 = \sum_{k=N+1}^{N(\lambda)} \left[\frac{1}{(\lambda_k(q_1)-\mu)^{m+1}} - \frac{1}{(\lambda_k(q_2)-\mu)^{m+1}}\right] \left(\int_\Gamma f\partial_\nu\varphi_k(q_1)\right)\partial_\nu\varphi_k(q_1),$$

$$I_2 = \sum_{k=N+1}^{N(\lambda)} \frac{1}{(\lambda_k(q_2)-\mu)^{m+1}} \left(\int_\Gamma f(\partial_\nu\varphi_k(q_1)-\partial_\nu\varphi_k(q_2))\right)\partial_\nu\varphi_k(q_1),$$

$$I_3 = \sum_{k=N+1}^{N(\lambda)} \frac{1}{(\lambda_{k,q_2}-\mu)^{m+1}} \left(\int_\Gamma f\partial_\nu\varphi_k(q_2)\right)\left[\partial_\nu\varphi_k(q_1)-\partial_\nu\varphi_k(q_2)\right].$$

- Using this decomposition, we estimate $\|\widetilde{\Lambda}_1^{(m)}(q_1,\mu)-\widetilde{\Lambda}_1^{(m)}(q_2,\mu)\|$. On the other hand, it is easy to prove

$$\|\widetilde{\Lambda}_2^{(m)}(q_1,\mu)-\widetilde{\Lambda}_2^{(m)}(q_2,\mu)\|\leq C\delta.$$

Putting these two estimates together, we obtain

$$\|\mathcal{R}(q_1,\lambda)-\mathcal{R}(q_2,\lambda)\|\leq C\rho^m\tau^{2(m+\frac{5}{4})}\delta.$$

- Finally, we arrive to the following estimate

$$C\|q\|_{L^2(\Omega)}^2\leq \frac{1}{\tau^{\frac{2}{n+2}}}+\tau^{\frac{2(n+1)}{n+2}}\left(\frac{1}{\rho^{2\sigma}}+\rho^{2m}\tau^{4m+5}\delta^2\right).$$

- $\rho=(2\Re\lambda)^\kappa$, for an appropriate choice of κ, gives

$$C\|q\|_{L^2(\Omega)}^2\leq \frac{1}{\tau^{\frac{2}{n+2}}}+\tau^{\frac{2(n+1)}{n+2}+4(\kappa+1)m+5}\delta^2.$$

- A standard minimization argument, with respect to τ, leads

$$C\|q\|_{L^2(\Omega)} \leq \delta^{\gamma},$$

with

$$\gamma = \frac{1}{n+2+2(n+2)(\kappa m + m + \frac{5}{4})}.$$

A stability estimate corresponding to the uniqueness result in Theorem 7 is given in following theorem.

Theorem 9

Let $q_1,\ q_2 \in L^\infty(\Omega)$ *such that* $q := q_1 - q_2 \in H_0^1(\Omega)$ *and*

$$\|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)} + \|q\|_{H_0^1(\Omega)} \leq c.$$

Fix $m > \frac{n}{2} + \frac{3}{4}$. *Let, for some* $\delta > 0$, $A > 0$,

$$|\lambda_k(q_1) - \lambda_k(q_2)| \leq \delta + Ak^{-\alpha},$$

$$k^{-\frac{2m}{n}+1}\|\partial_\nu \phi_k(q_1) - \partial_\nu \phi_k(q_2)\|_{L^2(\Gamma)} \leq \delta + Ak^{-\alpha},$$

with $\alpha > \frac{4m-1}{2n}$. *Then there exist* $C > 0$ *and* $0 < \gamma = \gamma(n, \alpha) < 1$ *such that*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\delta^\gamma.$$

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7 Extension to a magnetic Schödinger operator on compact Riemannian manifold

In this section $\psi_k(q) = \partial_\nu \varphi_k(q)$ on Γ, $k \geq 1$.

Lemma 1

For $q \in L^\infty(\Omega)$, $f \in H^{\frac{1}{2}}(\Gamma)$ and $\lambda \in \rho(A(q))$,

$$u(q,\lambda)(f) = \sum_{k\geq 1} \frac{1}{\lambda - \lambda_k(q)} \langle f, \psi_k(q)\rangle \varphi_k(q). \quad (3)$$

Moreover

$$\|u(q,\lambda)(f)\|^2_{L^2(\Omega)} = \sum_{k\geq 1} \frac{|\langle \psi_k(q), f\rangle|^2}{|\lambda_k(q) - \lambda|^2} \quad (4)$$

and

$$\|u(q,\lambda)(f)\|_{L^2(\Omega)} \to 0 \ \text{ as } \lambda \to -\infty. \quad (5)$$

Lemma 2

Let $q \in L^\infty(\Omega)$, $f \in H^{\frac{1}{2}}(\Gamma)$ and $\lambda, \mu \in \rho(A(q))$. If

$$u(q,\lambda,\mu)(f) := u(q,\lambda)(f) - u(q,\mu)(f),$$

then

$$\partial_\nu u(q,\lambda,\mu)(f) = \sum_{k\geq 1} \frac{\mu-\lambda}{(\mu-\lambda_k(q))\,(\lambda-\lambda_k(q))}\langle f,\psi_k(q)\rangle\psi_k(q). \tag{6}$$

Moreover, the series above converges in $H^{\frac{1}{2}}(\Gamma)$.

Proof. We firstly note that

$$u(q,\lambda,\mu)(f) = (\lambda-\mu)R(\lambda,q)[u(q,\mu)(f)].$$

Since the series in (3) converges in $L^2(\Omega)$, we derive

$$u(q,\lambda,\mu)(f) = (\lambda-\mu)\sum_{k\geq 1}\frac{1}{\mu-\lambda_k(q)}\langle f,\psi_k(q)\rangle R(\lambda,q)\varphi_k(q).$$

But

$$\sum_{k\geq 1}\frac{1}{\mu-\lambda_k(q)}\langle f,\psi_k(q)\rangle(A(q)-\lambda)R(\lambda,q)\varphi_k(q) = \sum_{k\geq 1}\frac{1}{\mu-\lambda_k(q)}\langle f,\psi_k(q)\rangle\varphi_k(q)$$

and series in right hand side converges in $L^2(\Omega)$. In other words, we proved that the series

$$\sum_{k\geq 1}\frac{1}{\lambda-\lambda_k(q)}\langle f,\psi_k(q)\rangle R(\lambda,q)\varphi_k(q) = \sum_{k\geq 1}\frac{1}{(\mu-\lambda_k(q))\,(\lambda-\lambda_k(q))}\langle f,\psi_k(q)\rangle\varphi_k(q).$$

converges in $H^2(\Omega)$. We complete the proof by using the continuity of the trace operator $w\to\partial_\nu w$ from $H^2(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$. $\square$

Lemma 3

Let $q_1, q_2 \in L^\infty(\Omega)$, $f \in H^{\frac{1}{2}}(\Gamma)$ *and* $\lambda \in \rho(A(q_1)) \cap \rho(A(q_2))$. *If*

$$u(q_1, q_2, \lambda)(f) := u(q_1, \lambda)(f) - u(q_2, \lambda)(f),$$

then

$$\|u(q_1, q_2, \lambda)(f)\|_{H^2(\Omega)} \to 0 \ \text{ as } \lambda \to -\infty,$$

implying

$$\|\partial_\nu u(q_1, q_2, \lambda)(f)\|_{H^{\frac{1}{2}}(\Gamma)} \to 0 \ \text{ as } \lambda \to -\infty. \tag{7}$$

Proof. We already saw that

$$u(q_1, q_2, \lambda)(f) = R(q_1, \lambda)\left[(q_2 - q_1)u(q_2, \lambda)\right].$$

Hence, by the resolvent estimate,

$$\begin{aligned} \|\lambda u(q_1, q_2, \lambda)(f)\|_{L^2(\Omega)} &\leq C\|u(q_1, q_2, \lambda)(f)\|_{L^2(\Omega)} \\ &\leq C\|u(q_2, \lambda)\|_{L^2(\Omega)}. \end{aligned}$$

Using that $u(q_1, q_2, \lambda)(f) \in H_0^1(\Omega)$ and

$$-\Delta u(q_1, q_2, \lambda)(f) = -q_1 u(q_1, q_2, \lambda)(f) + \lambda u(q_1, q_2, \lambda)(f) + (q_2 - q_1) u(q_2, \lambda),$$

we obtain from classical H^2 elliptic a priori estimate

$$\|u(q_1, q_2, \lambda)(f)\|_{H^2(\Omega)} \leq C \|u(q_2, \lambda)\|_{L^2(\Omega)}.$$

The lemma follows then from (5). □

Let $q_1, q_2 \in L^\infty(\Omega)$. For $j = 1, 2$, we have

$$\begin{aligned} S(q_j)(\lambda, \theta, \omega) &= \langle \Lambda(q_j, \lambda)\,(e_{\lambda,\omega})\,, \overline{e_{\lambda,\theta}} \rangle \\ &= \langle \partial_\nu u(q_j, \lambda, \mu)\,(e_{\lambda,\omega})\,, \overline{e_{\lambda,\theta}} \rangle + \langle \partial_\nu u(q_j, \mu) e_{\lambda,\omega}, \overline{e_{\lambda,\theta}} \rangle. \end{aligned}$$

Whence

$$\begin{aligned} S(q_1)(\lambda, \theta, \omega) - S(q_2)(\lambda, \theta, \omega) &= \langle \partial_\nu u(q_1, q_2, \mu) e_{\lambda,\omega}, \overline{e_{\lambda,\theta}} \rangle \\ &+ \langle [\partial_\nu u(q_1, \lambda, \mu) - \partial_\nu u(q_2, \lambda, \mu)]\,(e_{\lambda,\omega})\,, \overline{e_{\lambda,\theta}} \rangle. \end{aligned}$$

But

$$|\langle \partial_\nu u(q_1, q_2, \mu)\,(e_{\lambda,\omega})\,, \overline{e_{\lambda,\theta}} \rangle| \le C \|\partial_\nu u(q_1, q_2, \mu) e_{\lambda,\omega}\|_{L^2(\Gamma)}.$$

Hence, in light of (7),

$$\begin{aligned} & S(q_1)(\lambda, \theta, \omega) - S(q_2)(\lambda, \theta, \omega) \\ &= \lim_{\mu \to -\infty} \langle [\partial_\nu u(q_1, \lambda, \mu) - \partial_\nu u(q_2, \lambda, \mu)]\,(e_{\lambda,\omega})\,, \overline{e_{\lambda,\theta}} \rangle\,. \end{aligned} \tag{8}$$

To simplify our notations, we set

$$g(\psi) = \langle e_{\lambda,\omega}, \psi \rangle \langle \psi, \overline{e_{\lambda,\theta}} \rangle, \quad \psi \in L^2(\Gamma).$$

For $\psi_1, \psi_2 \in L^2(\Gamma)$, we have

$$|g(\psi_1) - g(\psi_2)| = |\langle e_{\lambda,\omega}, \psi_1 - \psi_2 \rangle \langle \psi_1, \overline{e_{\lambda,\theta}} \rangle + \langle e_{\lambda,\omega}, \psi_2 \rangle \langle \psi_2 - \psi_1, \overline{e_{\lambda,\theta}} \rangle|$$

Hence

$$|g(\psi_1) - g(\psi_2)| \leq C(| \langle \psi_1, \overline{e_{\lambda,\theta}} \rangle | + | \langle e_{\lambda,\omega}, \psi_2 \rangle |) \|\psi_1 - \psi_2\|_{L^2(\Gamma)}. \quad (9)$$

From now on, $\|q_j\|_{L^\infty(\Omega)} \leq \overline{m}$, $j = 1, 2$, where $\overline{m} > 0$ is a given constant. For $\Im\lambda \geq 1$ and $\mu \leq -(\overline{m} + 1)$, consider

$$f(\lambda, \mu) : \tau \in [-\overline{m}, \infty) \mapsto f(\lambda, \mu)(\tau) = \frac{\mu - \lambda}{(\lambda - \tau)(\mu - \tau)}.$$

From the mean value theorem, where $\tau_1, \tau_2 \geq -\overline{m}$, we get

$$\begin{aligned} &|f(\lambda,\mu)(\tau_1) - f(\lambda,\mu)(\tau_2)| \\ &\qquad \leq 2|\tau_1 - \tau_2| \max_{\tau\in[\tau_1,\tau_2]} \left[\frac{1}{(\lambda-\tau)^2} + \frac{1}{(\mu-\tau)^2}\right]. \end{aligned} \tag{10}$$

We have, for $j = 1, 2$, according to (6)

$$\langle \partial_\nu u(q_j,\lambda,\mu)(e_{\lambda,\omega}), \overline{e_{\lambda,\theta}}\rangle = \sum_{k\geq 1} f(\lambda,\mu)(\lambda_k(q_j))g(\psi_k(q_j)). \tag{11}$$

This identity in (8) yields

$$\begin{aligned} &S(q_1)(\lambda,\theta,\omega) - S(q_2)(\lambda,\theta,\omega) \\ &\qquad = \lim_{\mu\to-\infty} \left[\mathscr{S}_1(\lambda,\mu,\omega,\theta) + \mathscr{S}_2(\lambda,\mu,\omega,\theta)\right], \end{aligned} \tag{12}$$

with

$$\mathscr{S}_1(\lambda,\mu,\omega,\theta)=\sum_{k\geq 1}\left[f(\lambda,\mu)(\lambda_k(q_1))-f(\lambda,\mu)(\lambda_k(q_2))\right]g(\psi_k(q_1)),$$
$$\mathscr{S}_2(\lambda,\mu,\omega,\theta)=\sum_{k\geq 1}f(\lambda,\mu)(\lambda_k(q_2))\left[g(\psi_k(q_1))-g(\psi_k(q_2))\right].$$

Let $\overline{\delta}_0(q_1,q_2)=\sup_k|\lambda_k(q_1)-\lambda_k(q_2)|$. We get from (10)

$$\begin{aligned}&|[f(\lambda,\mu)(\lambda_k(q_1))-f(\lambda,\mu)(\lambda_k(q_2))]\,g(\psi_k(q_1))|\\ &\qquad\leq 2\overline{\delta}_0(q_1,q_2)\left[\frac{1}{(\lambda-\tau_k)^2}+\frac{1}{(\mu-\tau_k)^2}\right]|g(\psi_k(q_1))|,\end{aligned}$$

where τ_k is characterized by

$$\frac{1}{(\lambda-\tau_k)^2}+\frac{1}{(\mu-\tau_k)^2}=\max_{\tau\in[\lambda_k(q_1),\lambda_k(q_2)]}\left[\frac{1}{(\lambda-\tau)^2}+\frac{1}{(\mu-\tau)^2}\right].$$

We have

$$\frac{1}{(\lambda-\tau_k)^2}\leq\frac{2\overline{m}+1}{(\lambda-\lambda_k(q_1))^2}, \tag{13}$$

and

$$\frac{1}{(\mu-\tau_k)^2}\leq\frac{2\overline{m}+1}{(\mu-\lambda_k(q_1))^2}. \tag{14}$$

On the other hand

$$2\left|g(\lambda_k(q_1))\right|\leq\left|\langle e_{\lambda,\omega},\psi_k(q_1)\rangle\right|^2+\left|\langle\psi_k(q_1),\overline{e_{\lambda,\theta}}\rangle\right|^2 \tag{15}$$

A combination of (4) and (13)-(15) allows us to deduce that the series in $\mathscr{S}_1(\lambda,\mu,\omega,\theta)$ is absolutely convergent.

Note also that (14) entails

$$\frac{1}{(\mu-\tau_k)^2}\leq\frac{2\overline{m}+1}{(\overline{m}+1+\lambda_k(q_1))^2}. \tag{16}$$

Therefore the series $\mathscr{S}_1(\lambda,\mu,\omega,\theta)$ is absolutely uniformly convergent with respect to μ.

As a consequence of this fact, we obtain

$$\lim_{\mu\to-\infty}\mathscr{S}_1(\lambda,\mu,\omega,\theta)=\sum_{k\geq 1}\left[\frac{1}{\lambda-\lambda_k(q_1)}-\frac{1}{\lambda-\lambda_k(q_2)}\right]g(\psi_k(q_1)). \tag{17}$$

Next, since

$$\sup_{\{\mu\in(-\infty,-m-1],\ k\geq 1\}}\frac{|\lambda-\mu|}{|\mu-\lambda_k(q_2)|}=c(\lambda)<\infty,$$

$$|f(\lambda,\mu)(\lambda_k(q_2))\left[g(\psi_k(q_1))-g(\psi_k(q_2))\right]|$$
$$\leq C(\lambda)\frac{|\langle\psi_k(q_1),e_{\lambda,\omega}\rangle|+|\langle\psi_k(q_2),e_{\lambda,-\theta}\rangle|}{|\lambda-\lambda_k(q_2)|}\|\psi_k(q_1)-\psi_k(q_2)\|_{L^2((\Gamma)}.$$

But

$$\frac{1}{|\lambda-\lambda_k(q_2)|}\le c\frac{1}{|\lambda-\lambda_k(q_1)|},\quad k\ge 1.$$

Whence

$$|f(\lambda,\mu)(\lambda_k(q_2))\,[g(\psi_k(q_1))-g(\psi_k(q_2))]|$$
$$\le C(\lambda)\left(\frac{|\langle\psi_k(q_1),e_{\lambda,\omega}\rangle|}{|\lambda-\lambda_k(q_1)|}+\frac{|\langle\psi_k(q_2),e_{\lambda,-\theta}\rangle|}{|\lambda-\lambda_k(q_2)|}\right)\|\psi_k(q_1)-\psi_k(q_2)\|_{L^2(\Gamma)}.$$

We assume in the rest of this section

$$\delta_1:=\left(\sum_{k\ge 1}\|\psi_k(q_1)-\psi_k(q_2)\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}}<\infty,$$

Hence, from Cauchy-Schwarz inequality,

$$\sum_{k\ge 1}|f(\lambda,\mu)(\lambda_k(q_2))\,[g(\psi_k(q_1))-g(\psi_k(q_2))]|$$

$$\le C(\lambda)\left(\sum_{k\ge 1}\frac{|\langle\psi_k(q_1),e_{\lambda,\omega}\rangle|^2}{|\lambda-\lambda_k(q_1)|^2}+\sum_{k\ge 1}\frac{|\langle\psi_k(q_2),e_{\lambda,-\theta}\rangle|^2}{|\lambda-\lambda_k(q_2)|^2}\right)^{\frac{1}{2}}\delta_1.$$

That is that the series in $\mathscr{S}_2(\lambda,\mu,\omega,\theta)$ converges absolutely uniformly with respect to μ.Then

$$\lim_{\mu\to-\infty}\mathscr{S}_2(\lambda,\mu,\omega,\theta)=\sum_{k\geq1}\frac{1}{\lambda_k(q_2)-\lambda}\left[g(\psi_k(q_1))-g(\psi_k(q_2))\right]. \quad (18)$$

In light of (12), (17) and (18), we have the following identity

$$\begin{aligned} S(q_1)(\lambda,\theta,\omega) &- S(q_2)(\lambda,\theta,\omega) \quad (19)\\ &=\sum_{k\geq1}\left[\frac{1}{\lambda_k(q_1)-\lambda}-\frac{1}{\lambda_k(q_2)-\lambda}\right]g(\psi_k(q_1))\\ &+\sum_{k\geq1}\frac{1}{\lambda_k(q_2)-\lambda}\left[g(\psi_k(q_1))-g(\psi_k(q_2))\right] \end{aligned}$$

Introduce the following temporary notations

$$a_k(\lambda) = \left[\frac{1}{\lambda_k(q_1) - \lambda} - \frac{1}{\lambda_k(q_2) - \lambda}\right] g(\psi_k(q_1)),$$
$$b_k(\lambda) = \frac{1}{\lambda_k(q_2) - \lambda} \left[g(\psi_k(q_1)) - g(\psi_k(q_2))\right].$$

Fix an integer $N \geq 1$. From the preceding calculations

$$\sum_{k \geq N} |a_k| \leq C\delta_0 \left(\frac{|\langle \psi_k(q_1), e_{\lambda,\omega}\rangle|^2}{|\lambda_k(q_1) - \lambda|^2} + \frac{|\langle \psi_k(q_1), e_{-\lambda,\theta}\rangle|^2}{|\lambda_k(q_1) - \lambda|^2} \right)$$
$$\leq C\delta_0 \left(\|u(q_1, \lambda)(e_{\lambda,\omega})\|_{L^2(\Omega)} + \|u(q_1, \lambda)(e_{\lambda,-\theta})\|_{L^2(\Omega)} \right),$$

where

$$\delta_0 = \max_{k \geq N} |\lambda_k(q_1) - \lambda_k(q_2)|.$$

In light of the following lemma

Lemma 4

There exists a constant $C > 0$, depending only on n, Ω and $\overline{m}$ so that, for any $\lambda \in \mathbb{C}$ with $\Im\lambda \geq 1$ and $\omega \in \mathbb{S}^{n-1}$, we have

$$\|u(q_1, \lambda)(e_{\lambda,\omega})\|_{L^2(\Omega)} \leq C.$$

we get

$$\sum_{k \geq N} |a_k| \leq C\delta_0. \tag{20}$$

On the other hand, it is straightforward to check that

$$\lim_{|\lambda| \to \infty} \sum_{1 \leq k < N} |a_k| = 0. \tag{21}$$

Inequalities (20) and (21) entail

$$\limsup_{|\lambda|\to\infty} \sum_{k\geq 1} |a_k| \leq C\delta_0. \tag{22}$$

To estimate $\sum_k |b_k|$, we firstly note that

$$|b_k| \leq \left(\frac{|\langle \psi_k(q_1), e_{\lambda,\omega}\rangle|}{|\lambda - \lambda_k(q_2)|} + \frac{|\langle \psi_k(q_2), e_{\lambda,-\theta}\rangle|}{|\lambda - \lambda_k(q_2)|} \right) \|\psi_k(q_1) - \psi_k(q_2)\|_{L^2(\Gamma)}.$$

We have already seen

$$\frac{1}{|\lambda - \lambda_k(q_2)|} \leq C \frac{1}{|\lambda - \lambda_k(q_1)|}.$$

Hence

$$|b_k| \leq \left(\frac{|\langle \psi_k(q_1), e_{\lambda,\omega}\rangle|}{|\lambda - \lambda_k(q_2)|} + \frac{|\langle \psi_k(q_2), e_{\lambda,-\theta}\rangle|}{|\lambda - \lambda_k(q_2)|} \right) \|\psi_k(q_1) - \psi_k(q_2)\|_{L^2(\Gamma)}.$$

Applying Cauchy-Schwarz's inequality, we obtain

$$\left(\sum_{k\geq N}|b_k|\right)^2 \leq C\left(\sum_{k\geq N}\frac{|\langle\psi_k(q_1),e_{\lambda,\omega}\rangle|^2}{|\lambda-\lambda_k(q_2)|^2}+\sum_{k\geq N}\frac{|\langle\psi_k(q_2),e_{\lambda,-\theta}\rangle|^2}{|\lambda-\lambda_k(q_2)|^2}\right) \times\sum_{k\geq N}\|\psi_k(q_1)-\psi_k(q_2)\|^2_{L^2(\Gamma)}$$
$$\leq C\left(\|u(q_1,\lambda)(e_{\lambda,\omega})\|^2_{L^2(\Omega)}+\|u(q_2,\lambda)(e_{\lambda,-\theta})\|^2_{L^2(\Omega)}\right) \times\sum_{k\geq N}\|\psi_k(q_1)-\psi_k(q_2)\|^2_{L^2(\Gamma)}.$$

This inequality combined with Lemma 4 yields

$$\sum_{k\geq N}|b_k|\leq C\left(\sum_{k\geq N}\|\psi_k(q_1)-\psi_k(q_2)\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}}. \tag{23}$$

For $\epsilon>0$, there exists an integer N_ϵ so that

$$\sum_{k\geq N_\epsilon}\|\psi_k(q_1)-\psi_k(q_2)\|^2_{L^2(\Gamma)}\leq\epsilon^2.$$

$N = N_\epsilon$ in (23) yields

$$\sum_{k \geq N_\epsilon} |b_k| \leq C\epsilon \tag{24}$$

In a similar manner to $\sum_k |a_k|$, we can prove

$$\lim_{|\lambda| \to \infty} \sum_{1 \leq k < N_\epsilon} |b_k| = 0.$$

This and (24) give

$$\limsup_{|\lambda| \to \infty} \sum_{k \geq 1} |b_k| = 0. \tag{25}$$

We combine (2), (19), (24) and (25) in order to get

$$C\|q\|_{L^2(\Omega)} \leq \frac{1}{\tau^{\frac{1}{n+2}}} + \tau^{\frac{n}{2(n+2)}} \delta_0(q_1, q_2), \quad \tau > 1.$$

We derive from the preceding inequality, by minimizing with respect to τ,

Theorem 5

Let q_1, $q_2 \in L^\infty(\Omega)$ satisfying $q_1 - q_2 \in H_0^1(\Omega)$,

$$\|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)} + \|q_1 - q_2\|_{H_0^1(\Omega)} \leq c$$

and

$$\sum_{k\geq 1} \|\psi_k(q_1) - \psi_k(q_2)\|^2_{L^2(\Gamma)} < \infty$$

Then there exists $C = C(n, \Omega, c) > 0$ so that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\delta_0(q_1, q_2)^{\frac{1}{2n+2}},$$

where $\delta_0(q_1, q_2) = \sup_{k\geq N} |\lambda_k(q_1) - \lambda_k(q_2)|$.

Summary

1 The classical Borg-Levinson theorem

2 Nachman-Sylvester-Uhlmann's result

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7 Extension to a magnetic Schödinger operator on compact Riemannian manifold

Work in progress in collaboration with M. Bellassoued, D. Dos Santos Ferreira, Y. Kian and P. Stefanov.

. Consider (M, g) a compact Riemannian manifold with boundary Γ of dimension $n \geq 2$. We complexify the tangent and cotangent bundles as follows : $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ and $T_{\mathbb{C}}M^* = TM^* \otimes \mathbb{C}$. The metric tensor g gets extended as follows

$$g_{\mathbb{C}}(X_1 + iX_2, Y_1 + iY_2) = g(X_1, Y_1) - g(X_2, Y_2) + i[g(X_1, Y_2) + g(X_2, Y_1)].$$

In the sequel for sake of simplicity we drop the subscript $\mathbb{C}$ in $g_{\mathbb{C}}$.

The set of C^1 complex vector fields (resp. complex C^j 1-forms, $j = 1, 2$) over M is denoted by $V_1(M)$ (resp. $V_j^*(M)$, $j = 1, 2$). Unless otherwise stated, all the functions we consider are supposed complex-valued.

We adopt Einstein's summation convention. In the local coordinate system $x = (x^1, \ldots, x^n)$,

$$g(x) = g_{k\ell}(x)dx^k \otimes dx^\ell.$$

Denote by $|g|$ is the determinant of $(g^{k\ell})$, the inverse of the matrix $(g_{k\ell})$.

For $A = a_k dx^k \in V_1^*(M)$ real-valued 1-form, define the magnetic gradient ∇_A and magnetic divergence div_A as follows

$$\nabla_A u := g^{k\ell}(\partial_\ell u + i a_\ell u)\partial_k, \quad u \in C^1(M),$$
$$\mathrm{div}_A X := |g|^{-1/2}(\partial_k + i a_k)(|g|^{1/2} X^k), \quad X = X^\ell \partial_\ell \in V_1(M).$$

We call the operator Δ_A, given by

$$\Delta_A u = \mathrm{div}_A \nabla_A u = |g|^{-1/2}(\partial_k + i a_k)|g|^{1/2} g^{k\ell}(\partial_\ell u + i a_\ell), \quad u \in C^2(M),$$

the magnetic Laplace-Beltrami operator.

For $B = (A, V) \in V_1^*(M) \oplus L^\infty(M)$, define the magnetic Schrödinger operator $\mathcal{H}_B$ by

$$\mathcal{H}_B = -\Delta_A + V.$$

- Consider the unbounded operator $\mathcal{A}_B$ acting on $L^2(M)$ as follows

$$\mathcal{A}_B u = \mathcal{H}_B u, \quad D(\mathcal{A}_B) = \{u \in H_0^1(M);\ \Delta_A u \in L^2(M)\}.$$

Since $\mathcal{A}_B$ is self adjoint and has compact resolvent, its spectrum $\sigma(\mathcal{A}_B)$ consists in a sequence $\lambda_B = (\lambda_B^k)$ of (real) eigenvalues, counted according to their multiplicity, so that

$$-\infty < \lambda_B^1 \leq \lambda_B^2 \leq \ldots \leq \lambda_B^k \to +\infty \text{ as } k \to \infty.$$

In the sequel $\phi_B = (\phi_B^k)$ denotes an orthonormal basis of $L^2(M)$ consisting in eigenfunctions with ϕ_B^k associated to λ_B^k, for each k. Define

$$\psi_B^k = \langle \nabla_A \phi_B^k, \nu \rangle.$$

. Obstruction to uniqueness : Let $B = (A, q) \in V_1^*(M) \oplus L^\infty(M)$ and $\widetilde{B} = (A + d\chi, q)$, where $\chi \in C^2(M)$ is real valued. Then $\mathcal{H}_B = \mathcal{H}_{\widetilde{B}}$. If in addition $\chi = 0$ on Γ, then

$$(\lambda_B, \psi_B) = (\lambda_{\widetilde{B}}, \psi_{\widetilde{B}}).$$

. Inspired by the flat case, we seek $e^{i\sqrt{\lambda}\psi}b$ so that $e^{-i\sqrt{\lambda}\psi}(\Delta_A+\lambda)(e^{i\sqrt{\lambda}\psi}b)$ is independent on λ. This achieved whenever the phase ψ is the solution of the eikonal equation

$$|\nabla\psi|^2 = 1 \text{ in } M \tag{26}$$

and the amplitude b is a solution of the following transport equation

$$2\langle\nabla\psi, \nabla b\rangle + b\Delta\psi + 2i\langle A^\sharp, \nabla\psi\rangle b = 0 \text{ in } M. \tag{27}$$

The solvability of the eikonal equation (26) and equation (27) is possible when M is simple. That M is simply connected, any geodesic has no conjugate points and Γ is strictly convex, in the sense that the second fundamental form of the boundary is positive definite in every boundary point.

For $f \in H^{\frac{3}{2}}(\Gamma)$ and $\lambda \notin \sigma(\mathcal{A}_B)$, denote by $u_B(\lambda)(f) \in H^2(M)$ the unique solution of the BVP

$$\begin{cases} (\mathcal{H}_B - \lambda)u = 0 & \text{in } M, \\ u = f & \text{on } \Gamma. \end{cases} \tag{28}$$

For $\tau \geq 1$, set $\lambda_\tau = (\tau + i)^2$ and

$$\varphi_{\tau,b} = e^{i\sqrt{\lambda_\tau}\psi} b = e^{i(\tau+i)\psi} b,$$

where ψ and b satisfy respectively (26) and (27).

Define the family of D-to-N maps associated to B by

$$\Lambda_B(\lambda) : f \in H^{3/2}(\Gamma) \to \langle \nabla_A u_B(\lambda)(f), \nu \rangle \in H^{\frac{1}{2}}(\Gamma).$$

Let b_k be a solution of the transport equation (27) corresponding to $A = A_k$, $k = 1, 2$. Let $B_k = (A_k, V_k)$ set $\Lambda_k(\tau) = \Lambda_{B_k}(\lambda_\tau)$ and $u_k(\tau) = u_{B_k}(\lambda_\tau)$, $k = 1, 2$.

Define, for $k = 1, 2$,

$$S_k(\tau) = \int_\Gamma \Lambda_k(\tau)\varphi_{\tau,b_1}\overline{\varphi_{\tau,b_2}}d\sigma = \int_\Gamma \langle \nabla_{A_k} u_k(\tau)(f)\varphi_{\tau,b_1}, \nu\rangle \overline{\varphi_{\tau,b_2}}d\sigma.$$

. We know that $M \Subset M_1$ for some simple compact Riemannian manifold M_1.

In the sequel we assume that $A_1, A_2 \in V_2^*(M)$ with $A_1 = A_2$ in a neighborhood of Γ. We set $A = A_1 - A_2$ and $q = q_1 - q_2$ that we extend by 0 in $M_1 \setminus M$.

Let $\partial_+ SM_1 := \{(x,\theta) \in SM_1 : \ x \in \partial M_1, \ \langle \theta, \nu(x)\rangle < 0\}$. For $y \in \partial M_1$ and $\theta \in \partial_+ SM_1$, denote by $\tau_+(y,\theta)$ the time of existence in M_1 of the maximal geodesic $\gamma_{y,\theta}$ satisfying $\gamma_{y,\theta}(0) = y$ and $\gamma'_{y,\theta}(0) = \theta$.

Recall that the geodesic ray transform of the 1-form A is given by

$$\mathcal{I}_1 A(x,\theta) = \int_0^{\tau_+(x,\theta)} A(\gamma_{x,\theta}(s))\gamma'_{x,\theta}(s)ds, \quad (x,\theta) \in \partial_+ SM_1.$$

We construct the amplitudes b_1 and b_2 of the form, where $y \in \partial M_1$,

$$b_1(r,\theta) = h(\theta)\beta(r,\theta)^{-\frac{1}{4}} \exp\left(\mathrm{i} \int_0^{+\infty} \widetilde{A}_1(r+s,\theta)\theta ds \right),$$

$$b_2(r,\theta) = \beta(r,\theta)^{-\frac{1}{4}} \exp\left(-\mathrm{i} \int_0^{+\infty} \widetilde{A}_2(r+s,\theta)\theta ds \right).$$

for some $\widetilde{A}_1, \widetilde{A}_2 \in V_2^*(M_1)$, an arbitrary $h \in C^2(S_y M_1)$. Here β is the density of the volume form on $\exp_y^{-1}(M_1)$ in normal polar coordinates. Then

$$\lim_{\tau \to +\infty} \frac{S_1(\tau) - S_2(\tau)}{\tau} = 2i \int_{S_y^+(M_1)} \left(e^{i\mathcal{I}_1 A(y,\theta)} - 1 \right) h(\theta) d\theta \qquad (29)$$

Here $S_y^+(M_1) = \{\theta \in S_y M_1;\ \langle \theta, \nu(y) \rangle_{g(y)} < 0\}$.

Similarly, we have

$$\lim_{\tau \to +\infty} [S_1(\tau) - S_2(\tau)] = \int_{S_y^+(M_1)} \mathcal{I}_0 q(y,\theta) h(\theta) d\theta \qquad (30)$$

where $\mathcal{I}_0 q$ is the geodesic ray transform of q :

$$\mathcal{I}_0 q(x,\theta) = \int_0^{\tau_+(x,\theta)} q(\gamma_{x,\theta}(s)) ds, \quad (x,\theta) \in \partial_+ SM_1.$$

Assuming $q \in H_0^1(M)$ and, for some $s \in \left(0, \frac{1}{2}\right)$,

$$\sup k^{-\frac{s}{n}} |\lambda_k^1 - \lambda_k^2| + \sum \|\partial_\nu \varphi_k^1 - \partial_\nu \varphi_k^2\|_{L^2(\Gamma)} < \infty,$$

we prove

$$\lim_{\tau \to +\infty} [S_1(\tau) - S_2(\tau)] \leq \|h\|^2_{H^2(S_y^+ M_1)} \limsup |\lambda_k^1 - \lambda_k^2|. \qquad (31)$$

Take $h = \mathcal{I}_0 \mathcal{I}_0^* \mathcal{I}_0 q$ in (30) and (31), we get by using an interpolation inequality

$$C \|q\|^4_{L^2(M)} \leq \limsup |\lambda_k^1 - \lambda_k^2|.$$

Additionally, $A_1 = A_2$.