On the semi-stiff boundary conditions for the Ginzburg-Landau equations

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Two dimensional supraconductivity is described by the Ginzburg-Landau energy

\[ G_\epsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla u - iAu|^2 + \frac{1}{4\epsilon^2} \int_\Omega (1 - |u|^2)^2 + \frac{1}{2} \int_\Omega |\text{curl}A - h_{\text{ext}}|^2. \]
The full Ginzburg-Landau energy

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* \( \Omega \) is a smooth bounded connected domain.
* \( u : \Omega \to \mathbb{C} \) is the condensate wave function.
* \( A : \Omega \to \mathbb{R}^2 \) is the magnetic potential.
* \( h_{\text{ext}} \) is the external magnetic field.
* \( \epsilon = \frac{1}{\kappa} \) is the inverse of the G.L parameter.
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In their work F. Bethuel, H. Brézis and F. Hélein suggest to study the simplified G.L energy

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2$$

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subject to a Dirichlet condition \( g \in C^1(\partial A, S^1) \) with non-zero topological degree.
This model leads to quantized vortices as caused by a magnetic field in type II superconductors!
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The Euler-Lagrange equations are

$$\begin{cases} -\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0, & \text{in } \Omega, \\ |u| = 1, & \text{a.e on } \partial\Omega, \\ u \wedge \partial_{\nu} u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1)$$
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\end{cases} \quad (1)$$

In order to produce nonconstant solutions we can prescribe the degree on the connected components of $\partial\Omega$. 
Let $u \in H^1_2(\gamma, S^1)$, with $\gamma$ a simple, smooth, closed curve, the degree of $u$ on $\gamma$ is

$$\deg(u, \gamma) = \frac{1}{2\pi} \int_\gamma u \wedge \frac{\partial u}{\partial \tau} \, d\tau.$$
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The degree is an integer. The connected components of the space \( \mathcal{I} \) are classified using the degree. One can look solutions of (1) with prescribed degree(s) on \( \partial \Omega \).
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Problem : The degree is not continuous under weak \( H^1 \) convergence! Finding solutions of (1) with prescribed degree(s) on the boundary is a problem with lack of compactness.
Notations: If Ω is simply connected, let

\[ I_p = \{ u \in H^1(\Omega, \mathbb{R}^2); |u| = 1 \text{ on } \partial \Omega, \deg(u, \partial \Omega) = p \} \]

If Ω is doubly connected, Ω = ω₁ \ ω₀ with \( \overline{\omega_0} \subset \omega_1 \) let

\[ I_{p,q} = \{ u \in I; \deg(u, \partial \omega_1) = p, \deg(u, \partial \omega_0) = q \}. \]
Lemma (Price lemma)

Let \( \{u^{(n)}\} \subset I_{p,q} \) be a sequence which converges to \( u \) weakly in \( H^1(A, \mathbb{R}^2) \) with \( u \in I_{r,s} \). Then

\[
\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \leq \liminf_{n \to +\infty} \int_{\Omega} |\nabla u^{(n)}|^2 - \pi(|p - r| + |q - s|)
\]

or equivalently (by Sobolev embeddings)

\[
E_\epsilon(u) \leq \liminf_{n \to +\infty} E_\epsilon(u^{(n)}) - \pi(|p - r| + |q - s|). \tag{2}
\]
Let $m_\varepsilon(p) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, \deg(v, \partial\Omega) = p\}$ and $m_\varepsilon(p, q) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, \deg(v, \partial\omega_1) = p, \deg(v, \partial\omega_0) = q\}$.

**Lemma**

*Thanks to a special choice of test functions we have:*

\[
m_\varepsilon(p) \leq \pi|p|
\]

\[
m_\varepsilon(r, s) \leq \pi(|p - r| + |q - s|)
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**Lemma**

*Thanks to a special choice of test functions we have:*

\[ m_\varepsilon(p) \leq \pi |p| \]

\[ m_\varepsilon(r, s) \leq \pi (|p - r| + |q - s|) \]

**Proposition**

*Let $p \geq 1$ then $m_\varepsilon(p) = p\pi$ and is not attained.*
Now if we assume that $\Omega = \omega_1 \setminus \omega_0$, with $\omega_0 \subset \omega_1$ two smooth simply connected domain.

**Proposition**

If $p > 0 \geq q$ then $m_\varepsilon(p, q) = \pi(p + |q|)$ and is not attained.
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Existence/Nonexistence results for minimizing solutions
The case Ω doubly connected

Theorem (L. Berlyand, P. Mironescu, 2004)

1) If \( \text{cap}(\Omega) \geq \pi \) then \( m_\varepsilon(1, 1) \) is attained for all \( \varepsilon > 0 \).

2) If \( \text{cap}(\Omega) < \pi \) then there exists an \( \varepsilon_1 \) such that \( m_\varepsilon(1, 1) \) is attained for \( \varepsilon \geq \varepsilon_1 > 0 \) and \( m_\varepsilon(1, 1) \) is not attained for \( \varepsilon < \varepsilon_1 \).

Remark: \( \text{cap}(\Omega) \) is a measure of the thickness of \( \Omega \). For example, if \( \Omega = \{ z \in \mathbb{C}; \rho < |z| < R \} \) then

\[
\text{cap}(\Omega) = 2\pi \ln\left(\frac{R}{\rho}\right).
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Remark: \( \text{cap}(\Omega) \) is a measure of the thickness of \( \Omega \). For example if \( \Omega = \{z \in \mathbb{C}; \rho < |z| < R\} \) then \( \text{cap}(\Omega) = \frac{2\pi}{\ln(R/\rho)} \).
In order to prove the second part of the previous theorem one is lead to prove that:

\[ m_\infty(1, 1) = \inf \{ \int_\Omega |\nabla u|^2; u \in \mathcal{I}_{1,1} \} \]

is always attained ant that \( m_\infty(1, 1) < 2\pi \).
This suggest to study the problems $m_\infty(p, q)$, with $(p, q) \in \mathbb{Z}^2$.

**Remark**: Due to the conformal invariance of the Dirichlet integral one can assume that $\Omega = \{z \in \mathbb{C}; \rho < |z| < 1\}$.
This suggest to study the problems $m_\infty(p, q)$, with $(p, q) \in \mathbb{Z}^2$.

**Remark**: Due to the conformal invariance of the Dirichlet integral one can assume that $\Omega = \{z \in \mathbb{C}; \rho < |z| < 1\}$.

We are interested in critical points of $E_\infty(u) = \int_\Omega |\nabla u|^2$ in the space $\mathcal{I}$. They satisfy

\[
\begin{aligned}
-\Delta u &= 0, \quad \text{in } \Omega, \\
|u| &= 1, \quad \text{a.e. on } \partial \Omega, \\
u \wedge \partial_\nu u &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]
The Laplace equation with semi-stiff boundary conditions

Proposition

Let $p > 0 \geq q$ then $m_\infty(p, q) = \pi(p + |q|)$ and

* If $p > 0$ and $q = 0$ then there is no solution of (3) in $\mathcal{I}_{p,0}$.
* If $p > 0$ and $q < 0$ then there exists solutions of (3), all solutions are holomorphic and energy minimizing i.e $m_\infty(p, q)$ is attained.
Proposition

Let $p \geq 2$. There exists a sequence of critical radius $R_{cp}, R'_{cp}$ such that $0 = R_{c1} < R_{c2} < R_{c3} < ... < 1$, $0 = R'_{c1} < R'_{c2} < R'_{c3} < ... < 1$, $R_{cp} > R'_{cp}$ such that

1) If $\rho \geq R_{cp}$, then the minimum of $E_\infty$ in $I_{p, p}$ is attained, the minimizers are radial and $m_\infty(p, p) = 2\pi p \frac{1 - \rho^p}{1 + \rho^p}$.

2) If $\rho < R'_{cp}$ then the radial solutions of (3) is no longer minimizing.
Proposition

Let $p > 0$ there exists $c_p > 0$ and $\varepsilon_p > 0$ such that if $\text{cap}(\Omega) > c_p$ and $\varepsilon > \varepsilon_p$ then $m_\varepsilon(p, p)$ is attained.
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Application to G.L equations

Thank you for your attention!