On the semi-stiff boundary conditions for the Ginzburg-Landau equations

Rémy Rodiac

Université Paris-Est-Créteil

15th January 2014

Two dimensional supraconductivity is described by the Ginzburg-Landau energy

$$G_\epsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|u|^2)^2 + \frac{1}{2}\int_\Omega |curl A - h_{ext}|^2.$$

Two dimensional supraconductivity is described by the Ginzburg-Landau energy

$$G_\epsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|u|^2)^2 + \frac{1}{2}\int_\Omega |curlA - h_{ext}|^2.$$

* Ω is a smooth bounded connected domain.
* $u : \Omega \to \mathbb{C}$ is the condensate wave function.
* $A : \Omega \to \mathbb{R}^2$ is the magnetic potential.
* h_{ext} is the external magnetic field.
* $\varepsilon = \frac{1}{\kappa}$ is the inverse of the G.L parameter.

In this model $0 \leq |u|^2 \leq 1$ is the density of Cooper pairs of electrons.

In this model $0 \leq |u|^2 \leq 1$ is the density of Cooper pairs of electrons.
An important feature of the model is the existence of vortices. A vortex can be defined as small regions in the domain where $|u|^2$ is close to 0.

In this model $0 \leq |u|^2 \leq 1$ is the density of Cooper pairs of electrons.
An important feature of the model is the existence of vortices. A vortex can be defined as small regions in the domain where $|u|^2$ is close to 0.
The driving force for the appearing of such vortices is the magnetic field.

In their work F.Bethuel,H.Brézis and F.Hélein suggest to study the simplified G.L energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2$$

subject to a Dirichlet condition $g \in C^1(\partial A, \mathbb{S}^1)$ with non-zero topological degree.

In their work F.Bethuel,H.Brézis and F.Hélein suggest to study the simplified G.L energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2$$

subject to a Dirichlet condition $g \in C^1(\partial A, \mathbb{S}^1)$ with non-zero topological degree.
This model leads to quantized vortices as caused by a magnetic field in type II superconductors !

The Dirichlet boundary condition is not physical and it is natural to try to relax this condition.

The Dirichlet boundary condition is not physical and it is natural to try to relax this condition.
L.Berlyand and K.Voss propose to study critical points of $E_\varepsilon(u)$ in the space

$$\mathcal{I} = \{u \in H^1(\Omega;\mathbb{C}); |tr_{\partial\Omega} u| = 1 \ on\ \partial\Omega\}$$

The Dirichlet boundary condition is not physical and it is natural to try to relax this condition.
L.Berlyand and K.Voss propose to study critical points of $E_\varepsilon(u)$ in the space

$$\mathcal{I} = \{u \in H^1(\Omega;\mathbb{C}); |tr_{\partial\Omega} u| = 1 \ on \ \partial\Omega\}$$

The Euler-Lagarange equations are

$$\left\{ \begin{array}{rcll} -\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) & = & 0, & in \ \Omega, \\ |u| & = & 1, & a.e \ on \ \partial\Omega, \\ u \wedge \partial_\nu u & = & 0, & on \ \partial\Omega. \end{array} \right. \quad (1)$$

The Dirichlet boundary condition is not physical and it is natural to try to relax this condition.
L.Berlyand and K.Voss propose to study critical points of $E_\varepsilon(u)$ in the space

$$\mathcal{I} = \{u \in H^1(\Omega;\mathbb{C}); |tr_{\partial\Omega} u| = 1 \ on\ \partial\Omega\}$$

The Euler-Lagarange equations are

$$\begin{cases} -\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0, & in\ \Omega, \\ |u| = 1, & a.e\ on\ \partial\Omega, \\ u \wedge \partial_\nu u = 0, & on\ \partial\Omega. \end{cases} \quad (1)$$

In order to produce nonconstant solutions we can prescribe the degree on the connected components of $\partial\Omega$.

Let $u \in H^{\frac{1}{2}}(\gamma, \mathbb{S}^1)$, with γ a simple, smooth, closed curve, the degree of u on γ is

$$\deg(u, \gamma) = \frac{1}{2\pi} \int_\gamma u \wedge \frac{\partial u}{\partial \tau} d\tau.$$

Let $u \in H^{\frac{1}{2}}(\gamma, \mathbb{S}^1)$, with γ a simple, smooth, closed curve, the degree of u on γ is

$$\deg(u, \gamma) = \frac{1}{2\pi} \int_\gamma u \wedge \frac{\partial u}{\partial \tau} d\tau.$$

The degree is an integer. The connected components of the space $\mathcal{I}$ are classified using the degree. One can look solutions of (1) with prescribed degree(s) on $\partial\Omega$.

Let $u \in H^{\frac{1}{2}}(\gamma, \mathbb{S}^1)$, with γ a simple, smooth, closed curve, the degree of u on γ is

$$\deg(u, \gamma) = \frac{1}{2\pi} \int_\gamma u \wedge \frac{\partial u}{\partial \tau} d\tau.$$

The degree is an integer. The connected components of the space $\mathcal{I}$ are classified using the degree. One can look solutions of (1) with prescribed degree(s) on $\partial\Omega$.
Problem : The degree is not continuous under weak H^1 convergence ! Finding solutions of (1) with prescibed degree(s) on the boundary is a problem with **lack of compactness**.

Notations : If Ω is simply connected, let

$$\mathcal{I}_p = \{u \in H^1(\Omega, \mathbb{R}^2); |u| = 1 \ on\ \partial\Omega, deg(u, \partial\Omega) = p\}$$

If Ω is doubly connected, $\Omega = \omega_1 \setminus \omega_0$ with $\overline{\omega_0} \subset \omega_1$ let

$$\mathcal{I}_{p,q} = \{u \in \mathcal{I}; deg(u, \partial\omega_1) = p, deg(u, \partial\omega_0) = q\}.$$

Lemma (Price lemma)

Let $\{u^{(n)}\} \subset \mathcal{I}_{p,q}$ *be a sequence which converges to* u *weakly in* $H^1(A, \mathbb{R}^2)$ *with* $u \in \mathcal{I}_{r,s}$. *Then*

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx \leq \liminf_{n\to+\infty} \int_\Omega |\nabla u^{(n)}|^2 - \pi(|p-r| + |q-s|)$$

or equivalently (by sobolev embeddings)

$$E_\varepsilon(u) \leq \liminf_{n\to+\infty} E_\varepsilon(u^{(n)}) - \pi(|p-r| + |q-s|). \tag{2}$$

Let $m_\varepsilon(p) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, deg(v, \partial\Omega) = p\}$ and $m_\varepsilon(p,q) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, deg(v, \partial\omega_1) = p, deg(v, \partial\omega_0) = q\}$.

Lemma

Thanks to a special choice of test functions we have :

$$m_\varepsilon(p) \leq \pi|p|$$

$$m_\varepsilon(r,s) \leq \pi(|p-r| + |q-s|)$$

Let $m_\varepsilon(p) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, deg(v, \partial\Omega) = p\}$ and $m_\varepsilon(p, q) = \inf\{E_\varepsilon(v); v \in \mathcal{I}, deg(v, \partial\omega_1) = p, deg(v, \partial\omega_0) = q\}$.

Lemma

Thanks to a special choice of test functions we have :

$$m_\varepsilon(p) \leq \pi|p|$$

$$m_\varepsilon(r, s) \leq \pi(|p - r| + |q - s|)$$

Proposition

Let $p \geq 1$ then $m_\varepsilon(p) = p\pi$ and is not attained.

Now if we assume that $\Omega = \omega_1 \setminus \omega_0$, with $\omega_0 \subset \omega_1$ two smooth simply connected domain.

Proposition

If $p > 0 \geq q$ then $m_\varepsilon(p, q) = \pi(p + |q|)$ and is not attained.

Theorem (L.Berlyand,P.Mironescu,2004)

1) *If cap*$(\Omega) \geq \pi$ *then* $m_\varepsilon(1,1)$ *is attained for all* $\varepsilon > 0$.
2) *If cap*$(\Omega) < \pi$ *then there exists an* ε_1 *such that* $m_\varepsilon(1,1)$ *is attained for* $\varepsilon \geq \varepsilon_1 > 0$ *and* $m_\varepsilon(1,1)$ *is not attained for* $\varepsilon < \varepsilon_1$.

Theorem (L.Berlyand,P.Mironescu,2004)

1) *If* $cap(\Omega) \geq \pi$ *then* $m_\varepsilon(1,1)$ *is attained for all* $\varepsilon > 0$.
2) *If* $cap(\Omega) < \pi$ *then there exists an* ε_1 *such that* $m_\varepsilon(1,1)$ *is attained for* $\varepsilon \geq \varepsilon_1 > 0$ *and* $m_\varepsilon(1,1)$ *is not attained for* $\varepsilon < \varepsilon_1$.

Remark : $cap(\Omega)$ is a measure of the thickness of Ω. For example if $\Omega = \{z \in \mathbb{C}; \rho < |z| < R\}$ then $cap(\Omega) = \frac{2\pi}{\ln(R/\rho)}$.

In order to prove the second part of the previous theorem one is lead to prove that :

$$m_\infty(1,1) = \inf\{\int_\Omega |\nabla u|^2; u \in \mathcal{I}_{1,1}\}$$

is always attained ant that $m_\infty(1,1) < 2\pi$.

This suggest to study the problems $m_\infty(p, q)$, with $(p, q) \in \mathbb{Z}^2$.
Remark : Due to the conformal invariance of the Dirichlet integral one can assume that $\Omega = \{z \in \mathbb{C}; \rho < |z| < 1\}$.

This suggest to study the problems $m_\infty(p,q)$, with $(p,q)\in\mathbb{Z}^2$.
Remark : Due to the conformal invariance of the Dirichlet integral one can assume that $\Omega=\{z\in\mathbb{C};\rho<|z|<1\}$.

We are interested in critical points of $E_\infty(u)=\int_\Omega|\nabla u|^2$ in the space $\mathcal{I}$. They satisfy

$$\begin{cases} -\Delta u = 0, & in\ \Omega, \\ |u| = 1, & a.e\ on\ \partial\Omega, \\ u\wedge\partial_\nu u = 0, & on\ \partial\Omega. \end{cases} \tag{3}$$

Proposition

Let $p > 0 \geq q$ then $m_\infty(p, q) = \pi(p + |q|)$ and

* *If $p > 0$ and $q = 0$ then there is no solution of (3) in $\mathcal{I}_{p,0}$.*
* *If $p > 0$ and $q < 0$ then there exists solutions of (3), all solutions are holomorphic and energy minimizing i.e $m_\infty(p, q)$ is attained.*

Proposition

Let $p \geq 2$. There exists a sequence of critical radius R_{c_p}, R'_{c_p} with $0 = R_{c_1} < R_{c_2} < R_{c_3} < ... < 1$, $0 = R'_{c_1} < R'_{c_2} < R'_{c_3} < ... < 1$, $R_{c_p} > R'_{c_p}$ such that

1) *If $\rho \geq R_{c_p}$, then the minimum of E_∞ in $\mathcal{I}_{p,p}$ is attained, the minimizers are radial and $m_\infty(p,p) = 2\pi p \frac{1-\rho^p}{1+\rho^p}$.*
2) *If $\rho < R'_{c_p}$ then then the radial solutions of (3) is no longer minimizing.*

Proposition

Let $p > 0$ there exists $c_p > 0$ and $\varepsilon_p > 0$ such that if $cap(\Omega) > c_p$ and $\varepsilon > \varepsilon_p$ then $m_\varepsilon(p, p)$ is attained.

Thank you for your attention !