The ground state of a system of two coupled Gross-Pitaevskii equations in the Thomas-Fermi limit.

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$$
(S_\varepsilon) \qquad \begin{cases} \varepsilon^2 \Delta \eta_1 + \left(\mu_1 - |x|^2\right) \eta_1 - 2\alpha_1 \eta_1^3 - 2\alpha_0 \eta_2^2 \eta_1 = 0 \\ \varepsilon^2 \Delta \eta_2 + \left(\mu_2 - |x|^2\right) \eta_2 - 2\alpha_2 \eta_2^3 - 2\alpha_0 \eta_1^2 \eta_2 = 0, \end{cases}
$$

where $x \in \mathbb{R}^d$, $d = 1, 2, 3$, ε is a small parameter, $\mu_1, \mu_2 > 0$ are chemical potentials, $\alpha_0, \alpha_1, \alpha_2$ are positive constants.

Goal

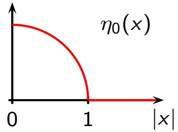
- construct $(\eta_{1\varepsilon}, \eta_{2\varepsilon})$ solution to (S) such that $\eta_{1\varepsilon}, \eta_{2\varepsilon} > 0$,
- understand the behaviour of $(\eta_{1\varepsilon}, \eta_{2\varepsilon})$ as $\varepsilon \to 0$ (Thomas–Fermi limit).

Results for one single equation

(GP) $$\varepsilon^2 \Delta \eta_\varepsilon + (1 - |x|^2)\eta_\varepsilon - \eta_\varepsilon^3 = 0, \quad x \in \mathbb{R}^d,$$

Ground states η_ε of (GP) converge in the limit $\varepsilon \to 0$ to

$$\eta_0(x) = \begin{cases} \sqrt{1 - |x|^2} & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$



We set $\eta_\varepsilon(x) = \varepsilon^{1/3}\gamma(y)$ where $y = \frac{1-|x|^2}{\varepsilon^{2/3}}$.

Then $\gamma = \gamma_0 + \varepsilon^{2/3}\gamma_1 + \varepsilon^{4/3}\gamma_2 + \cdots$, where

- γ_0 is the Hastings-McLeod solution of the Painlevé II equation, that is the unique solution of
  (PII) $$4\gamma_0'' + y\gamma_0 - \gamma_0^3 = 0, \quad y \in \mathbb{R}.$$
  that satisfies $\gamma_0(y) \underset{y\to+\infty}{\sim} \sqrt{y}$, $\gamma_0(y) \underset{y\to-\infty}{\to} 0$,
- for $n \geqslant 1$, γ_n satisfies $\gamma_n(y) \underset{y\to\pm\infty}{\longrightarrow} 0$.

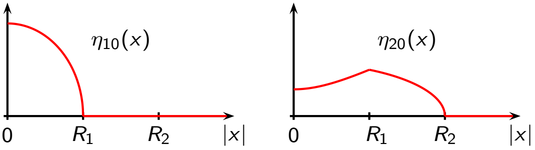
The Thomas–Fermi limit for the system

We expect
$$(\eta_{1\varepsilon}, \eta_{2\varepsilon}) \underset{\varepsilon\to 0}{\longrightarrow} (\eta_{10}, \eta_{20}),$$
where η_{10}, η_{20} are compactly supported in disks with radius R_1 and R_2, and given by (we assume $R_1 < R_2$)

	$0 \leqslant \lvert x\rvert \leqslant R_1$	$R_1 \leqslant \lvert x\rvert \leqslant R_2$	$\lvert x\rvert \geqslant R_2$
η_{10}^2	$\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}(R_1^2 - \lvert x\rvert^2)$	0	0
η_{20}^2	$\frac{R_2^2-R_1^2}{2\alpha_2} + \frac{\Gamma_1}{2\alpha_2\Gamma_{12}}(R_1^2 - \lvert x\rvert^2)$	$\frac{R_2^2-\lvert x\rvert^2}{2\alpha_2}$	0

It implies $\Gamma_2/\Gamma_{12} > 0$. We will assume later $\Gamma_2 > 0$, $\Gamma_{12} > 0$.

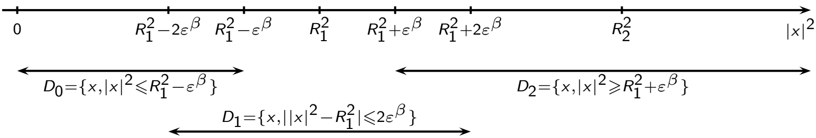
We also assume $R_2^2 > \frac{\alpha_0}{\alpha_1}\frac{\Gamma_2}{\Gamma_{12}} R_1^2$ (ensures that $\eta_{20}(0) > 0$).



Different ansatz depending on $x \in \mathbb{R}^d$

Let $\beta \in (0, 2/3)$.

$\mathbb{R}^d$ is divided in three domains D_0, D_1, D_2 defined as follows.



In each of these domains, we use a different variable and a different ansatz to look for a solution (η_1, η_2) of (S_ε) that converges to (η_{10}, η_{20}) as $\varepsilon \to 0$.

domain	D_0	D_1	D_2
variable	$z = R_1^2 - \|x\|^2$	$y_1 = \frac{R_1^2 - \|x\|^2}{\varepsilon^{2/3}}$	$y_2 = \frac{R_2^2 - \|x\|^2}{\varepsilon^{2/3}}$
ansatz	$\eta_1(x) = \omega(z)$ $\eta_2(x) = \tau(z)$	$\eta_1(x) = \varepsilon^{1/3}\nu(y_1)$ $\eta_2(x) = \varepsilon^{1/3}\lambda(y_1)^{1/2}$	$\eta_1(x) = 0$ $\eta_2(x) = \varepsilon^{1/3}\mu(y_2)$

Formal expansions of ω and τ in D_0

$x \in D_0$: $\quad \eta_1(x) = \omega(z), \quad \eta_2(x) = \tau(z), \quad z = R_1^2 - |x|^2.$

$$\omega = \omega_0 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + \cdots$$

$$\tau = \tau_0 + \varepsilon^2\tau_1 + \varepsilon^4\tau_2 + \cdots ,$$

where

$$\omega_0(z) = \eta_{10}(x) = \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}\right)^{1/2} z^{1/2},$$

$$\tau_0(z) = \eta_{20}(x) = \left(\frac{R_2^2 - R_1^2}{2\alpha_2} + \frac{\Gamma_1}{2\alpha_2\Gamma_{12}} z\right)^{1/2}$$

For $m \geqslant 1$, (ω_m, τ_m) are obtained recursively by plugging these expansions into (S_ε) and cancelling the terms corresponding to successive powers of ε^2 in the equations.

Formal Expansion of μ in D_2

$x \in D_2$: $\eta_1(x) = 0$, $\eta_2(x) = \varepsilon^{1/3}\mu(y_2)$, $y_2 = \frac{R_2^2 - |x|^2}{\varepsilon^{2/3}}$

$$\mu = \mu_0 + \varepsilon^{2/3}\mu_1 + \varepsilon^{4/3}\mu_2 + \cdots$$

Plugging the expansion in (S_ε), we find for every n:

$$\mu_n(y_2) = \frac{R_2^{1/3}}{(2\alpha_2)^{1/2}} R_2^{-4n/3} \gamma_n\left(\frac{y_2}{R_2^{2/3}}\right),$$

where the functions γ_n are the one which where obtained in the expansion of the ground state of the single equation.

Formal expansions of ν and λ in D_1

$x \in D_1$: $\eta_1(x) = \varepsilon^{1/3}\nu(y_1)$, $\eta_2(x) = \varepsilon^{1/3}\lambda(y_1)^{1/2}$, $y_1 = \frac{R_1^2 - |x|^2}{\varepsilon^{2/3}}$

$$\nu = \nu_0 + \varepsilon^{2/3}\nu_1 + \varepsilon^{4/3}\nu_2 + \cdots$$
$$\lambda = \lambda_{-1}\varepsilon^{-2/3} + \lambda_0 + \varepsilon^{2/3}\lambda_1 + \varepsilon^{4/3}\lambda_2 + \cdots$$

Plugging these expansions into (S_ε), we obtain:

$$\lambda_{-1}(y_1) = \frac{R_2^2 - R_1^2}{2\alpha_2}$$

$$4R_1^2\nu_0'' + \Gamma_2 y_1 \nu_0 - 2\alpha_1\Gamma_{12}\nu_0^3 = 0.$$

$$\lambda_0(y_1) = \frac{y_1}{2\alpha_2} - \frac{\alpha_0}{\alpha_2}\nu_0(y_1)^2.$$

For $n \geqslant 1$,
$$\Big(-4R_1^2\partial_{y_1}^2 + \underbrace{6\alpha_1\Gamma_{12}\nu_0^2 - \Gamma_2 y_1}_{=:W(y_1)>0}\Big)\nu_n = F_n,$$

$$\lambda_n = -2\frac{\alpha_0}{\alpha_2}\nu_0\nu_n + \frac{2\alpha_2}{(R_2^2 - R_1^2)^2}\delta_n,$$

where F_n and δ_n can be explicitely written in terms of the λ_k's and the ν_k's for $0 \leqslant k \leqslant n-1$.

Calculation of ν_0

- For $R_1 < |x| < R_2$, $y_1 \to -\infty$ and $\varepsilon^{1/3}\nu(y_1) \underset{\varepsilon\to 0}{\longrightarrow} 0$, thus

$$\nu_0(y_1) \underset{y_1\to-\infty}{\longrightarrow} 0.$$

- For $|x| < R_1$, $y_1 \to +\infty$ and $\varepsilon^{1/3}\nu_0(y_1) \underset{\varepsilon\to 0}{\longrightarrow} \left(\frac{\Gamma_2}{2\alpha_1\Gamma_{12}}(R_1^2 - |x|^2)\right)^{1/2}$, thus

$$\nu_0(y_1) \underset{y_1\to+\infty}{\sim} \left(\frac{\Gamma_2 y_1}{2\alpha_1\Gamma_{12}}\right)^{1/2}.$$

Moreover,

$$4R_1^2\nu_0'' + \Gamma_2 y_1\nu_0 - 2\alpha_1\Gamma_{12}\nu_0^3 = 0$$

has a non-trivial solution with fast decay at $-\infty$ only if $\Gamma_2 > 0$ and $\Gamma_{12} > 0$. Under this condition,

$$\nu_0(y_1) = \frac{R_1^{1/3}\Gamma_2^{1/3}}{(2\alpha_1)^{1/2}\Gamma_{12}^{1/2}}\gamma_0\left(\frac{\Gamma_2^{1/3}y_1}{R_1^{2/3}}\right).$$

Truncation of the asymptotic expansions

From now on,

$$\omega(z) = \sum_{m=0}^{M} \varepsilon^{2m}\omega_m(z), \quad \tau(z) = \sum_{m=0}^{M} \varepsilon^{2m}\tau_m(z),$$
$$\nu(y_1) = \sum_{n=0}^{N} \varepsilon^{2n/3}\nu_n(y_1), \quad \lambda(y_1) = \sum_{n=-1}^{N} \varepsilon^{2n/3}\lambda_n(y_1),$$
$$\mu(y_2) = \sum_{n=0}^{L} \varepsilon^{2n/3}\mu_n(y_2).$$

Comparison of (ω, τ) and $\varepsilon^{1/3}(\nu, \lambda^{1/2})$ in $D_0 \cap D_1$

Lemma. Let $N > 0$, $M \geqslant \frac{\beta}{2-3\beta} N$. Then for $l = 0, 1, 2$,

$$\left\| \frac{d^l}{dz^l} \left(\omega(z) - \varepsilon^{1/3}\nu(y_1) \right) \right\|_{L^\infty(D_0 \cap D_1)} = o(\varepsilon^{\beta(N-1/2-l)}),$$

$$\left\| \frac{d^l}{dz^l} \left(\tau(z) - \varepsilon^{1/3}\lambda(y_1)^{1/2} \right) \right\|_{L^\infty(D_0 \cap D_1)} = o(\varepsilon^{\beta(N-l)}).$$

Comparison of $\varepsilon^{1/3}(\nu, \lambda^{1/2})$ and $(0, \varepsilon^{1/3}\mu)$ in $D_1 \cap D_2$

Lemma. Let $N > 0$, $L \geqslant 3\beta(N+1)/2$. Then for $l \in \{0, 1, 2\}$,

$$\forall \alpha > 0, \quad \left\| \frac{d^l}{dz^l} \left(\varepsilon^{1/3} \nu(y_1) \right) \right\|_{L^\infty(D_1 \cap D_2)} = o(\varepsilon^\alpha)$$

and

$$\left\| \frac{d^l}{dz^l} \left(\varepsilon^{1/3} \lambda(y_1)^{1/2} - \varepsilon^{1/3} \mu(y_2) \right) \right\|_{L^\infty(D_1 \cap D_2)} = o(\varepsilon^{\beta(N+1-l)}).$$

Approximate solution of (S_ε)

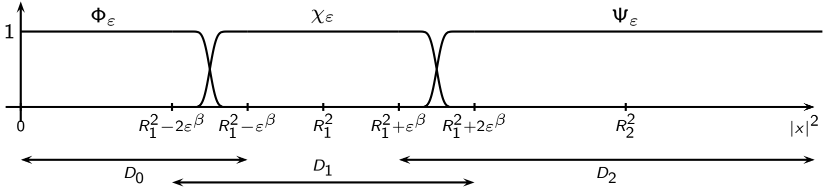
We plug into (S_ε) the ansatz

$$\eta_1 = \varepsilon^{1/3}\left(\rho_1 + \varepsilon^{2(N+1)/3} P\right),$$

$$\eta_2 = \varepsilon^{1/3}\left(\rho_2 + \varepsilon^{2(N+1)/3} Q\right),$$

where

$$\varepsilon^{1/3}\rho_1 = \Phi_\varepsilon \omega + \chi_\varepsilon \varepsilon^{1/3} \nu,$$
$$\varepsilon^{1/3}\rho_2 = \Phi_\varepsilon \tau + \chi_\varepsilon \varepsilon^{1/3} \lambda^{1/2} + \Psi_\varepsilon \varepsilon^{1/3} \mu.$$



The equation satisfied by (P, Q)

$$A_\varepsilon \begin{bmatrix} P \\ Q \end{bmatrix} = f_\varepsilon^0(x) + f_\varepsilon^2(x, P, Q) + f_\varepsilon^3(x, P, Q),$$

where

$$A_\varepsilon = \begin{bmatrix} -\varepsilon^{4/3}\Delta + p_\varepsilon(x) & r_\varepsilon(x) \\ r_\varepsilon(x) & -\varepsilon^{4/3}\Delta + q_\varepsilon(x) \end{bmatrix},$$

$$p_\varepsilon(x) = -\frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}} - y_1 + 6\alpha_1\rho_1^2 + 2\alpha_0\rho_2^2,$$

$$q_\varepsilon(x) = -y_2 + 6\alpha_2\rho_2^2 + 2\alpha_0\rho_1^2, \qquad r_\varepsilon(x) = 4\alpha_0\rho_1\rho_2,$$

$$f_\varepsilon^0(x) = \varepsilon^{-2(N+1)/3}\begin{bmatrix} \varepsilon^{4/3}\Delta\rho_1 + \frac{\alpha_0}{\alpha_2}\frac{R_2^2 - R_1^2}{\varepsilon^{2/3}}\rho_1 + y_1\rho_1 - 2\alpha_1\rho_1^3 - 2\alpha_0\rho_2^2\rho_1 \\ \varepsilon^{4/3}\Delta\rho_2 + y_2\rho_2 - 2\alpha_2\rho_2^3 - 2\alpha_0\rho_1^2\rho_2 \end{bmatrix},$$

$$f_\varepsilon^2(x, P, Q) = -2\varepsilon^{2(N+1)/3}\begin{bmatrix} 3\alpha_1\rho_1 P^2 + 2\alpha_0\rho_2 PQ + \alpha_0\rho_1 Q^2 \\ 3\alpha_2\rho_2 Q^2 + 2\alpha_0\rho_1 PQ + \alpha_0\rho_2 P^2 \end{bmatrix},$$

$$f_\varepsilon^3(x, P, Q) = -2\varepsilon^{4(N+1)/3}\begin{bmatrix} \alpha_1 P^3 + \alpha_0 PQ^2 \\ \alpha_2 Q^3 + \alpha_0 P^2 Q \end{bmatrix}.$$

Estimate on the source term f_ε^0

Let $N \gg 1$. Then there exists $\beta \in (0, 2/3)$, L, M large integers such that

$$\left\| f_\varepsilon^0 \right\|_{L^2(\mathbb{R}^d)^2} \lesssim \varepsilon^{-2}.$$

Estimate on the resolvent of A_ε

A_ε is invertible, and

$$\| A_\varepsilon^{-1} \|_{\mathcal{L}(L^2(\mathbb{R}^d)^2, H_w^1(\mathbb{R}^d)^2)} \lesssim \varepsilon^{-4/3},$$

where $H_w^1(\mathbb{R}^d)^2$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R}^d)^2$ for the norm

$$\begin{aligned} \|(P,Q)\|^2_{H_w^1(\mathbb{R}^d)^2} \quad = \quad & \int_{\mathbb{R}^d} \left(|\nabla P|^2 + |\nabla Q|^2 \right) dx \\ & + \int_{\mathbb{R}^d} \max(1, \min(|y_1|, |y_2|))(|P|^2 + |Q|^2) dx. \end{aligned}$$

The fixed point argument

The fixed point theorem is applied to the map

$$\begin{array}{rcl} \Theta_\varepsilon \; : \; H^1_w(\mathbb{R}^d)^2 & \longrightarrow & H^1_w(\mathbb{R}^d)^2 \\ (P,Q) & \longrightarrow & A_\varepsilon^{-1} f_\varepsilon^0 + A_\varepsilon^{-1} f_\varepsilon^2(P,Q) + A_\varepsilon^{-1} f_\varepsilon^3(P,Q) \end{array}$$

in the ball $\mathcal{B}_R$ of $H^1_w(\mathbb{R}^d)^2$ centered at the origin, with radius

$$R = 2\left\| A_\varepsilon^{-1} f_\varepsilon^0 \right\|_{H^1_w(\mathbb{R}^d)^2} \lesssim \varepsilon^{-10/3}.$$

From the Sobolev embedding $H^1_w(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset L^4(\mathbb{R}^d)$ we infer: for every $(P,Q) \in H^1_w(\mathbb{R}^d)^2$, we have $f_\varepsilon^2 \in L^2(\mathbb{R}^d)^2$ and $\|f_\varepsilon^2(P,Q)\|_{L^2(\mathbb{R}^d)^2} \lesssim \varepsilon^{2N/3+1/3} \|(P,Q)\|^2_{H^1_w(\mathbb{R}^d)^2}$.

Thus, $\quad \left\| A_\varepsilon^{-1} f_\varepsilon^2(P,Q) \right\|_{H^1_w(\mathbb{R}^d)^2} \lesssim \varepsilon^{2N/3-1} \|(P,Q)\|^2_{H^1_w(\mathbb{R}^d)^2}$.

$H^1_w(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset L^6(\mathbb{R}^d)$ implies

$$\left\| A_\varepsilon^{-1} f_\varepsilon^3(P,Q) \right\|_{H^1_w(\mathbb{R}^d)^2} \lesssim \varepsilon^{4N/3} \|(P,Q)\|^3_{H^1_w(\mathbb{R}^d)^2}.$$

It follows that the ball is stable by Θ_ε, and similar arguments show that it is a contraction on that ball.

The main result

Theorem Let $N \gg 1$. Then there exists $\beta \in (0, 2/3)$, L, M large integers, such that for ε small enough, there is $(P, Q) \in (H^1_w)^2$ with

$$\|(P, Q)\|_{(H^1_w)^2} \lesssim \varepsilon^{-10/3}$$

such that

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \Phi_\varepsilon \omega + \chi_\varepsilon \varepsilon^{1/3} \nu + \varepsilon^{2N/3+1} P \\ \Phi_\varepsilon \tau + \chi_\varepsilon \varepsilon^{1/3} \lambda^{1/2} + \Psi_\varepsilon \varepsilon^{1/3} \mu + \varepsilon^{2N/3+1} Q \end{pmatrix}$$

solves (S_ε).