ON LATTICES WITH FINITE RENORMALIZED COULOMBIAN INTERACTION ENERGY IN THE PLANE

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ABSTRACT. We present criteria for the coulombian interaction energy of infinitely many points in the plane with a uniformly charged background introduced in [5] to be finite, as well as examples. We also show that in this unbounded setting, it is not always possible to project an L^2_{loc} vector field onto the set of gradients in a way that reduces its average L^2 norm on large balls.

1. INTRODUCTION

In [5], S.Serfaty and the second author introduced an energy describing the coulombian interaction energy of infinitely many unit positive charges in the plane with a uniformly negatively charged background. This energy was dubbed W after a similar energy which arose in [13] as a sharp (codimension 2) interface limit of a vector Modica-Mortola type functional, in a bounded domain.

In turns out that W, defined below, appears naturally in several variational settings, where minimizers or quasi-minimizers exhibit a vortex structure. This is the case of the Ginzburg-Landau model of superconductivity ([5]) or the Ohta-Kawasaki model for diblock co-polymers ([11]), other natural candidates would be superfluids modeled by the Gross-Pitaevski functional or certain models for dislocations where the analysis of vortices is already well advanced ([14, 15, 16] for example, the literature on these subjects is already quite large).

In all these contexts, there exist limits when simultaneously the characteristic size of a vortex goes to zero (a sharp codimension 2 interface limit) and the number of vortices tends to $+\infty$, because of an external applied field or boundary conditions. Then the vortices of minimizers will be described at the macro-scale by a certain optimal density and at the micro-scale by discrete subsets of $\mathbb{R}^2$ which minimize W, this energy accounting for the interaction of individual vortices with the field generated by the density, the latter being constant at the microscale if the optimal density is well-behaved.

Another context in which W appears is the case of weighted Fekete N-sets. The Fekete N-sets are N-tuples of points in the plane which minimize an energy w_n which is the sum of their logarithmic interaction plus a potential term which confines them. In the aforementionned models, the Fekete N sets would arise when the sharp interface limit is taken but the number of vortices remains equal to N. When N tends to $+\infty$, again W governs the arrangement of the points at the microscale ([6], [7]).

Finally, the energy W plays a role in the Coulomb gas model in statistical mechanics, for which the probability law density of N particles is $P_N = \frac{1}{Z_{n,\beta}} e^{-\beta w_n}$, where β is the inverse of temperature and $Z_{n,\beta}$ is the partition function. There ([6], [7]) the minimum of W appears in the asymptotic expansion of $\log Z_{N,\beta}$ as $N \to +\infty$ for large β, i.e. small temperature.

Until now, some basic but usefal facts are known about W (see [5]): It is bounded below, admits minimizers, and minimizers may be approximated by doubly periodic configurations of points. It is also known that among perfect (Bravais) lattices, the triangular lattice is the unique minimizer of W ([5]). The minimal value of W is not known, even though it can be used to express other quantities as in the aforementionned expansion of $logZ_{N,\beta}$, but also the energy of weighted Fekete N-sets (see [17]). Finding the minimum of W seems to be a challenging problem, even though such results exist for energies that similarly measure the distance of a discrete subset of $\mathbb{R}^2$ to the uniform measure ([18]).

In this paper we focus on the natural question of which discrete subsets $\Lambda \subset \mathbb{R}^2$ are such that $W(\Lambda) < +\infty$. This question is of interest for instance in the context of Coulomb-gases where it is shown in [6] that for any β, one can extract in a natural way from P_N and as $N \to +\infty$ a probability measure on discrete subsets of $\mathbb{R}^2$ which is concentrated on those sets for which W is finite.

It turns out that even though the logarithmic potential involved in W is very long range, a lot can be said about this question. It turns out also that it connects to problems of independent interest and which to our knowledge have not been adressed in the litterature. The first one is that of a Hodge decomposition for L^2_{loc} instead of L^2 vector fields. More precisely is it possible to project an L^2_{loc} vector field on the set of gradients in a way that reduces its average L^2 norm on large balls ? We provide a counter-example in Remark 3. The second question is that of estimating the number of points of a quasi-cristalline lattice in a ball of radius R, a classical problem for Bravais lattices (see [2] for instance). Although we have no answer to this, our results give a hint that such estimates might exist.

2. Main results

Let us now define W. We note that somewhat different definition was recently introduced in [12] which generalizes to other dimensions. It is shown in [12] that both definitions agree in the case of Λ's for which the distance between two distinct points of Λ is bounded below by a positive number, what we denote below *uniform* Λ's.

Given a discrete set Λ in the plane and a real number $m \geq 0$, the renormalized energy introduced in [5] is defined in several steps.

First, denoting $\nu := \sum_{p\in\Lambda} \delta_p$ and for any vector-field j solving

$$-\text{div}(j) = 2\pi(\nu - m) \text{ in } \mathbb{R}^2, \tag{1}$$

and belonging to $L^2_{\text{loc}}(\mathbb{R}^2 \setminus \Lambda, \mathbb{R}^2)$ we define $W(j)$ as follows: For any $R > 1$ we denote by χ_R a smooth approximation of the indicator function of B_R, the ball centered at 0 with radius R. More precisely we assume that

$$\chi_R \geq 0,\ \|\nabla \chi_R\|_\infty \leq C,\ \chi_R \equiv 1 \text{ on } B_{R-1} \text{ and } \chi_R \equiv 0 \text{ on } \mathbb{R}^2 \setminus B_R, \tag{2}$$

where C is independent of R. Then we let

$$W(j) = \limsup_{R\to\infty} \frac{W(j,\chi_R)}{|B_R|}, \quad W(j,\chi_R) = \limsup_{\eta\to 0} \frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda} B(p,\eta)} \chi_R|j|^2 + \pi\log\eta \sum_{p\in\Lambda}\chi_R(p) \tag{3}$$

Second, we consider the set $\mathcal{F}_\Lambda$ of vector fields in $L^2_{\rm loc}(\mathbb{R}^2\setminus\Lambda,\mathbb{R}^2)$ satisfying (1) for a given Λ and m, and the subset $\mathcal{P}_\Lambda$ of curl-free vector fields in $\mathcal{F}_\Lambda$, or equivalently the set of those elements in $\mathcal{F}_\Lambda$ which are gradients. We may now define

$$W(\Lambda):=\inf_{\nabla U\in\mathcal{P}_\Lambda}W(\nabla U),\quad \tilde W(\Lambda):=\inf_{j\in\mathcal{F}_\Lambda}W(j). \tag{4}$$

Note that $\mathcal{F}_\Lambda$ and $\mathcal{P}_\Lambda$ depend on m, hence so do $W(\Lambda)$ and $\tilde W(\Lambda)$. But in fact (see below) the value of m is determined by Λ in the sense that $W(\Lambda)$ or $\tilde W(\Lambda)$ can only be finite for at most one value of m (which is the asymptotic density of Λ whenever it exists). In any case, the value of m will always be clear from the context or made precise.

Remark 1. *It will be useful to generalize somewhat the above definition to allow j's satisfying* (1) *with*

$$\nu:=\sum_{p\in\Lambda}\alpha_p\delta_p.$$

In this case one should modifiy the definition of $W(j,\chi_R)$:

$$W(j,\chi_R):=\limsup_{\eta\to 0}\frac12\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta)}\chi_R|j|^2+\pi|\alpha_p|^2\log\eta\sum_{p\in\Lambda}\chi_R(p). \tag{5}$$

In [5], only W is considered. One could think at first that W and $\tilde W$ are equal, the argument being the following: Since $W(j)$ may be seen as the average of $|j|^2$ over $\mathbb{R}^2$ (with the infinite part due to the Dirac masses in (1) removed), then projecting onto the set of curl-free fields would reduce this quantity, so that the infimum of $W(j)$ over $\mathcal{F}_\Lambda$ would in fact be acheived by some $j\in\mathcal{P}_\Lambda$, proving that $W(\Lambda)=\tilde W(\Lambda)$. It turns out however that this is not the case and in fact we prove (see Theorem 1 below) that with $m=0$,

Theorem. *$W(\mathbb{N})=+\infty$ and $\tilde W(\mathbb{N})<+\infty$.*

The rest of the paper is devoted to giving sufficient conditions on Λ for $\tilde W$ and/or W to be finite. There are roughly two factors which can make W or $\tilde W$ infinite. First, there is the logarithmic interaction between pairs of points, which can be made infinite by bringing points very close to each other: we will not consider this factor here and to rule it out we restrict ourself to *uniform* Λ's in the following sense.

Definition 1. *Given a discrete set Λ and weights $\{\alpha_p\}_{p\in\Lambda}$, we say that*

$$\nu=2\pi\sum_{p\in\Lambda}\alpha_p\delta_p$$

is of uniform type *if*

$$\min_{p\neq q\in\Lambda}|p-q|>0,\quad \sup_{p\in\Lambda}|\alpha_p|<\infty.$$

If the weights are all equal to 1 we simply say Λ is of uniform type.

The second factor which can make W or $\tilde W$ infinite is the interaction with the background. If we restrict ourselves to uniform Λ's, then for a given m the quantities $W(\Lambda)$ or $\tilde W(\Lambda)$ measure how close $\sum_{p\in\Lambda}\delta_p$ is to a uniform density m. Our second main result shows that this can be measured by simply counting the number of points of Λ in any given ball (see Theorems 2 and 5). In particular we have

Theorem. *Assume that Λ is uniform and that there exists $m, C \geq 0$ and $\varepsilon \in (0,1)$ such that for any $x \in \mathbb{R}^2$ and $R > 1$ we have, denoting $\sharp E$ the number of elements in E,*

$$\left|\sharp\left(B(x,R) \cap \Lambda\right) - m\pi R^2\right| \leq CR^{1-\varepsilon} \tag{6}$$

Then $W(\Lambda) < +\infty$ for this value of m.

This criterion for finiteness is optimal in the sense that if we replace the right-hand side in (6) by $CR^{1+\varepsilon}$, then it is not difficult to construct Λ's satisfying (6) and having infinite renormalized energies (see Proposition 5). This criterion can be relaxed a bit in the case of $\tilde{W}$ (see Theorem 5).

This leaves open the case $\varepsilon = 0$ (in which case $\mathbb{N}$ and $\mathbb{Z}$ satisfy (6) with $m = 0$). In this case we are able to prove a partial result (see Theorem 6 for a variant)

Theorem. *Let $A \subset \mathbb{Z}^2$ and $\Lambda := \mathbb{Z}^2 \setminus A$. Assume there exists some constant $C > 0$ such that for all $x \in \mathbb{R}^2$ and $R > 1$ we have*

$$\sharp\left(A \cap B(x,R)\right) \leq CR.$$

Then $\tilde{W}(\Lambda) < +\infty$, with $m = 1$.

The proof of this theorem is based on the fact (see Proposition 6) that under the above hypothesis there exists a bijection between $\mathbb{Z}^2 \setminus A$ and $\mathbb{Z}^2$ under which points are moved at uniformly bounded distances. This is a discrete analogue of a result of G.Strang [10]. Its conclusion cannot be improved to $W(\Lambda) < +\infty$, see Proposition 4.

The criterion $\left|\sharp\left(B(x,R) \cap \Lambda\right) - m\pi R^2\right| \leq CR^{1-\varepsilon}$ is satisfied by perfect (or Bravais) lattices, or more generally by doubly periodic lattices (see [3]) — even though in this case (see below) the conclusion of Theorem 2 is almost trivial. However we are not aware that this is known for quasi-cristalline lattices, and thus we give a construction similar to that of Theorem 2 which allows us to conclude for an exemple of Penrose-type lattice Λ that $\tilde{W}(\Lambda) < +\infty$. We have not sought generality in this direction, and refer to Section 7 for the construction of Λ and the proof that $\tilde{W}(\Lambda)$ is finite.

3. Some properties of W, $\tilde{W}$

We always assume the following property of $\nu := \sum_{p\in\Lambda} \delta_p$, which is satisfied in particular if Λ is uniform.

$$\limsup_{R\to+\infty} \frac{\nu(B_R)}{|B_R|} < +\infty. \tag{7}$$

We begin by recalling some facts from [5, 6].

- **Structure of $\mathcal{P}_\Lambda$:** If Λ satisfies (7) and $W(\Lambda)$ is finite, then the set $\{\nabla U \in \mathcal{P}_\Lambda \mid W(\nabla U) < +\infty\}$ is a 2-dimensional affine space. Any two gradients in this set differ by a constant vector.
- **Minimization:** For any given m, the function $\Lambda \to W$ defined over the set of Λ's satisfying (7) is bounded from below and admits a minimizer.
- **Scaling:** Denote W_m the renormalized energy with background $m \in \mathbb{R}$. If j satisfies (1) and (7) holds, then

$$W(j) = m\left(W(j') - \frac{\pi}{2}\log m\right), \quad \text{with} \quad j'(\cdot) = \frac{1}{\sqrt{m}} j\left(\frac{\cdot}{\sqrt{m}}\right).$$

Cutoffs: If (7) holds, then the value of $W(\Lambda)$ (or $\tilde{W}(\Lambda)$) does not depend on the particular choice of cut-off functions χ_R as long as they satisfy the stated properties.

Perfect lattices: Assume $\Lambda = \mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$ where $(\vec{u}, \vec{v})$ is a basis of $\mathbb{R}^2$ satisfying the normalized volume condition $|\vec{u} \wedge \vec{v}| = 1$. Let Λ^* be the dual lattice of Λ. Then, taking $m = 1$,

$$W(\Lambda) = \pi \lim_{x \to 0} \left(\sum_{p \in \Lambda^* \setminus \{0\}} \frac{e^{2i\pi p \cdot x}}{4\pi^2 |p|^2} + \log |x| \right) - \frac{\pi}{2} \log 2\pi.$$

Moreover, the minimum of W among lattices of this type is acheived by the triangular lattice

$$\Lambda_1 := \sqrt{\frac{2}{\sqrt{3}}} \left((1,0)\mathbb{Z} \oplus \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \mathbb{Z} \right).$$

Uniqueness of m: For a given Λ, there can be at most one value of m for which $W(\Lambda) < +\infty$. Indeed if j_1 (resp. j_2) satisfy (1) with m_1 (resp. m_2) then $-\mathrm{div}(j_1 - j_2) = m_2 - m_1$, and if $m_1 \neq m_2$ this implies that $W(j_1)$ and $W(j_2)$ cannot both be finite. To see this one can use Proposition 1 below in case the points in Λ are uniformly spaced. Otherwise one has to resort to the corresponding result in [5].

One of the main points in [5, 6] is the fact that W is bounded below. This is in fact very easy to prove in the case of Λ's — or more generally ν's — which are of uniform type. It is a consequence of the following useful fact.

Proposition 1. *If j satisfies* (1) *with ν of uniform type, then for any $\delta < \frac{1}{2} \inf_{p \neq q \in \Lambda} |p - q|$ there exists $g : \mathbb{R}^2 \to \mathbb{R}$ and $C > 0$ such that*

$$g \geq -C, \tag{8}$$

such that

$$g = \frac{1}{2}|j|^2, \quad \text{on} \quad \mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \delta), \tag{9}$$

and such that for any compactly supported lipschitz function χ,

$$\left| \int_{\mathbb{R}^2} \chi g - W(j, \chi) \right| \leq CN \|\nabla \chi\|_\infty, \tag{10}$$

where $N = \sharp\{p \in \Lambda \mid B(p, \delta) \cap \mathrm{Supp}\, \nabla \chi \neq \varnothing\}$.

Remark 2. *Note that if we take χ such that $\chi = 1$ on $B(p, \delta)$ and $\chi = 0$ on every other $B(q, \delta)$ for $q \neq p \in \Lambda$ then* (10) *implies that $\int_{\mathbb{R}^2} \chi g = W(j, \chi)$. This implies in particular, approximating the indicator function $\mathbf{1}_{B(p,\delta)}$ by such functions, that for any $p \in \Lambda$*

$$\int_{B(p,\delta)} g = W(j, \mathbf{1}_{B(p,\delta)}) \tag{11}$$

Proof. In $\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \delta)$, we let $g = \frac{1}{2}|j|^2$. Then, for any $p \in \Lambda$ and any $r \in (0, \delta)$ such that $|j| \in L^2(\partial B(p, r))$ — this is the case for a.e. r — we define $\lambda_{p,r} > 0$ to be a value of λ such that

$$\frac{1}{2} \int_{\partial B(p,r)} \min(|j|^2, \lambda) = \frac{\pi \alpha_p^2}{r} - 2\pi^2 \alpha_p m r. \tag{12}$$

The fact that $\lambda_{p,r}$ is well defined follows from the fact that the left-hand side of (12) is a continuous increasing function of λ which increases from 0 to (as $\lambda \to +\infty$)

$$\frac{1}{2}\int_{\partial B(p,r)} |j|^2 \geq \frac{1}{4\pi r}\left(\int_{\partial B(p,r)} j\cdot\nu\right)^2 = \frac{\pi}{r}\left(\alpha_p - m\pi r^2\right)^2 \geq \frac{\pi\alpha_p^2}{r} - 2\pi^2\alpha_p m r.$$

For any $r \leq \delta$ we let, on $\partial B(p,r)$,

$$g := \frac{1}{2}(|j|^2 - \lambda_{p,r})_+ - \pi\alpha_p m - \frac{\alpha_p^2}{\delta^2}\log\frac{1}{\delta}.$$

Then (9) is obviously satisfied, and (8) is satisfied with

$$C = \left(\sup_{p\in\Lambda}\alpha_p\right)^2 \frac{|\log\delta|}{\delta^2} + \pi|m|\sup_{p\in\Lambda}|\alpha_p|.$$

It remains to prove (10). For any function χ and any $\eta \leq \delta$ we have

$$\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta)} \chi\left(\frac{|j|^2}{2} - g\right) = \sum_{p\in\Lambda}\int_{B(p,\delta)\setminus B(p,\eta)} \chi\left(\frac{|j|^2}{2} - g\right). \tag{13}$$

Then, writing A for the annulus $B(p,\delta)\setminus B(p,\eta)$,

$$\int_A \chi\left(\frac{|j|^2}{2} - g\right) = \chi(p)\int_A\left(\frac{|j|^2}{2} - g\right) + \int_A (\chi - \chi(p))\left(\frac{|j|^2}{2} - g\right). \tag{14}$$

We have for any $r \leq \delta$, on $\partial B(p,r)$

$$\left(\frac{|j|^2}{2} - \frac{1}{2}(|j|^2 - \lambda_{p,r})_+\right) = \frac{1}{2}\min\left(|j|^2, \lambda_{p,r}\right),$$

hence using (12) we find

$$\begin{aligned}\int_A\left(\frac{|j|^2}{2} - g\right) &= \int_\eta^\delta \frac{\pi\alpha_p^2}{r} - 2\pi^2\alpha_p m r\,dr + \pi(\delta^2 - \eta^2)\left(\pi\alpha_p m + \frac{\alpha_p^2}{\delta^2}\log\frac{1}{\delta}\right) \\ &= \pi\alpha_p^2\log\frac{1}{\eta} - \pi\eta^2\frac{\alpha_p^2}{\delta^2}\log\frac{1}{\delta}.\end{aligned} \tag{15}$$

On the other hand, since $|\chi - \chi(p)| \leq r\|\nabla\chi\|_\infty$, and using (12) we have

$$\left|\int_A (\chi - \chi(p))\left(\frac{|j|^2}{2} - g\right)\right| \leq \|\nabla\chi\|_\infty \int_\eta^\delta \frac{r}{2}\int_{\partial B(p,r)} \left(\min\left(|j|^2, \lambda_{p,r}\right) + C\right) dr \leq C\|\nabla\chi\|_\infty.$$

This together with (13),(14) and (15) yields

$$\left|\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta)} \chi\left(g - \frac{|j|^2}{2}\right) - \sum_{p\in\Lambda}\pi\chi(p)\alpha_p^2\log\eta\right| \leq CN\|\nabla\chi\|_\infty,$$

where $N = \sharp\{p\in\Lambda \mid B(p,\delta)\cap \operatorname{Supp}\nabla\chi \neq \varnothing\}$. This proves (10). □

Note that, contrary to the corresponding result in [5], we have not proved that the constant C in (8) is universal, which is a delicate point. We have included this weaker result for the sake of self-containedness and because it has a simple proof.

4. Examples with finite or infinite energy.

We begin by showing that moving the points in $\mathbb{Z}^2$ at a bounded distance yields a Λ with finite energy, assuming Λ is uniform.

Proposition 2. *Let Λ satisfy $\inf_{x,y\in\Lambda,x\neq y}|x-y|>0$ and let $\Phi:\Lambda\to\mathbb{Z}^2$ be a bijective map such that $\sup_{p\in\Lambda}|\Phi(p)-p|<\infty$. Then $\tilde W(\Lambda)<+\infty$, with $m=1$.*

Proof. Let $R_1=2\sup_{p\in\Lambda}|\Phi(p)-p|$. Then for every $p\in\Lambda$, we solve

$$\begin{cases} -\triangle U_p &= 2\pi\left(\delta_p-\delta_{\Phi(p)}\right) \quad \text{in } B(p,R_1)\\ \frac{\partial U_p}{\partial\nu} &= 0 \quad \text{on } \partial B(p,R_1)\end{cases}$$

where ν is the outer unit normal on the boundary. Let V be the $\mathbb{Z}^2$-periodic solution — which is unique modulo an additive constant — of

$$-\triangle V = 2\pi\left(\sum_{p\in\mathbb{Z}^2}\delta_p-1\right)\quad\text{in } \mathbb{R}^2$$

Then by periodicity $|V(x)+\log|x-p||$ is bounded in $C^2\left(\cup_{p\in\mathbb{Z}^2}B(p,1/4)\right)$, while $V(x)$ is bounded in C^2 of the complement. More precisely we have the (see for instance [5])

$$V(x)=\sum_{p\in\mathbb{Z}^2\setminus\{0\}}\frac{e^{2i\pi p\cdot x}}{2\pi|p|^2}$$

Now we define $j:\mathbb{R}^2\to\mathbb{R}$ by

$$j=\nabla V+\sum_{p\in\Lambda}\nabla U_p,$$

where ∇U_p is is extended by 0 outside of $B(p,R_1)$ and thus defined on the whole of $\mathbb{R}^2$.

From the assumptions on Λ and Φ the sum above is finite on any compact set and thus j is well defined and solves

$$-\mathrm{div}(j) = 2\pi\left(\sum_{p\in\Lambda}\delta_p-1\right)\quad\text{in } \mathbb{R}^2.$$

On the other hand, $U_p(x)+\log|x-p|-\log|x-\Phi(p)|$ is bounded in $C^2(B(p,R_1))$, uniformly with respect to $p\in\Lambda$. It follows that $j+\nabla\log|x-p|$ is bounded in $B(p,\delta)$ uniformly with respect to $p\in\Lambda$, and j is bounded in $\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\delta)$, where $\delta>0$ is half the minimal distance between points of Λ. A straightforward consequence is that $W(j)<+\infty$ and then $\tilde W(\Lambda)<+\infty$. □

We will prove below that the conclusion in the above proposition cannot be improved to $W(\Lambda)<+\infty$.

A consequence of Proposition 2 is

Corollary 1. *We have, with $m=1$,*

$$\tilde W(\mathbb{Z}^2\setminus\mathbb{Z})<\infty,\ \tilde W(\mathbb{Z}^2\setminus\mathbb{N})<\infty$$

Proof. We construct a bijective map from $\Phi : \mathbb{Z}^2 \setminus \mathbb{Z} \to \mathbb{Z}^2$ by

$$\Phi(p_1, p_2) = \begin{cases} (p_1, p_2 - 1) & \text{if } p_2 \geq 1 \\ (p_1, p_2) & \text{if } p_2 < 0 \end{cases}$$

The desired result follows from the above proposition. The proof for $\mathbb{Z}^2 \setminus \mathbb{N}$ is similar. □

A second tool for constructing j's with finite energy is

Proposition 3. *Assume j_1 (resp. j_2) satisfy* (1) *with a ν_1 (resp. ν_2) of uniform type. Assume also that ν_1 and ν_2 satisfy* (7) *and that $\nu_1 + \nu_2$ is of uniform type.*

Then, if $W(j_1) < \infty$ for a background m_1 and $W(j_2) < \infty$ for the background m_2, we have

$$W(j_1 + j_2) < \infty \text{ for the background } m_1 + m_2.$$

First, we prove two lemmas.

Lemma 1. *Assume j satisfies* (1) *and* (7) *with ν of uniform type, and assume $W(j) < \infty$. Then there exists some positive constant C depending on j such that for any $R > 1$ and $\delta < \frac{1}{2} \inf\{|p - q| \mid p \neq q \in \Lambda\}$,*

$$\int_{B_R \setminus \cup_{p \in \Lambda} B(p,\delta)} |j|^2 \leq CR^2, \quad \int_{B_R \cap \cup_{p \in \Lambda} B(p,\delta)} |j - G|^2 \leq CR^2,$$

where $G(x) := \alpha_p \dfrac{x - p}{|x - p|^2}$ if $x \in B(p, \delta)$ with $p \in \Lambda$.

Proof. Let g be constructed in Proposition 1. From (10), we have

$$\int \chi_R g \leq W(j, \chi_R) + Cn(R) \leq CR^2,$$

where $n(R) := \sharp(\Lambda \cap B_{R+1})$. Hence

$$\int \chi_R g \leq CR^2. \tag{16}$$

On the other hand, since $g \geq -C$ and from the properties of χ_R, we have

$$\int \chi_R g \geq \int_{B_R} g - CR \geq \int_{B_R \setminus \cup_{p \in \Lambda} B(p,\delta)} \frac{1}{2}|j|^2 + \sum_{p \in \Lambda, B(p,\delta) \subset B_R} \int_{B(p,\delta)} g - CR. \tag{17}$$

For any $p \in \Lambda$, we define

$$W(j, \mathbf{1}_{B(p,\delta)}) := \limsup_{\eta \to 0} \frac{1}{2} \int_{B(p,\delta) \setminus B(p,\eta)} |j|^2 + \pi \alpha_p^2 \log \eta$$

We have, denoting $A = B(p, \delta) \setminus B(p, \eta)$,

$$\begin{aligned} \frac{1}{2} \int_A |j|^2 &= \frac{1}{2} \int_A |G|^2 + |j - G|^2 + 2G \cdot (j - G) \\ &= \pi \alpha_p^2 \log \frac{\delta}{\eta} + \frac{1}{2} \int_A |j - G|^2 + \alpha_p \int_\eta^\delta \frac{dr}{r} \int_{\partial B(p,r)} \nu \cdot (j - G) \\ &= \pi \alpha_p^2 \log \frac{\delta}{\eta} + \frac{1}{2} \int_A |j - G|^2 + \alpha_p \int_\eta^\delta \frac{dr}{r} \int_{B(p,r)} \operatorname{div}(j - G) \\ &= \pi \alpha_p^2 \log \frac{\delta}{\eta} + \frac{1}{2} \int_A |j - G|^2 + \pi^2 \alpha_p m(\delta^2 - \eta^2). \end{aligned}$$

Hence, we obtain

$$W(j,\mathbf{1}_{B(p,\delta)})=\limsup_{\eta\to 0}\frac{1}{2}\int_A |j|^2+\pi\alpha_p^2\log\eta=\pi\alpha_p^2\log\delta+\frac{1}{2}\int_{B(p,\delta)}|j-G|^2+\pi^2\alpha_p m\delta^2 \tag{18}$$

Thus, using (11),

$$\int_{B(p,\delta)} g=\pi\alpha_p^2\log\delta+\frac{1}{2}\int_{B(p,\delta)}|j-G|^2+\pi^2\alpha_p m\delta^2 \tag{19}$$

Gathering (16) to (19), we get

$$CR^2\ge\int\chi_R g\ge\int_{B_R\setminus\cup_{p\in\Lambda}B(p,\delta)}\frac{1}{2}|j|^2+\sum_{p\in\Lambda,B(p,\delta)\subset B_R}\frac{1}{2}\int_{B(p,\delta)}|j-G|^2-CR^2.$$

This gives the desired result. □

Lemma 2. *Assume j satisfies (1) and (7) with ν of uniform type and let G be the function defined in Lemma 1 — for some $\delta<\frac{1}{2}\inf\{|p-q|\mid p\neq q\in\Lambda\}$ — and extended by 0 on $\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\delta)$. Then*

$$W(j)<\infty\ \Leftrightarrow\ \limsup_{R\to\infty}\fint_{B_R}|j-G|^2<\infty$$

where $\fint_A$ denotes the average over A.

Proof. The " $\Longrightarrow$ " part of the assertion follows from Lemma 1. We prove the reverse implication. We denote by g the result of applying Proposition 1 to j.

Then from the properties of χ_R and using (10), (8),

$$W(j,\chi_R)\le\int_{B_R}g\chi_R+CR^2\le CR^2+\int_{B_R\setminus\cup_{p\in\Lambda}B(p,\delta)}g+\sum_{p\in\Lambda\cap B_R}\int_{B(p,\delta)}g.$$

Then, as in the proof of Lemma 1,

$$\int_{B(p,\delta)}g=W(j,\mathbf{1}_{B(p,\delta)})=\frac{1}{2}\int_{B(p,\delta)}|j-G|^2+O(1).$$

Using this and (9) we find

$$W(j,\chi_R)\le CR^2+\frac{1}{2}\int_{B_{R+\delta}}|j-G|^2.$$

This yields the desired result. □

Proof of Proposition 3. We denote Λ_i the discrete set related to j_i for $i=1,2$ and Λ one related to j_1+j_2. We write $\nu_i=\sum_{p\in\Lambda_i}\alpha_{i,p}\delta p$, $i=1,2$. Then we choose

$$\delta<\frac{1}{2}\min\left(\inf\{|p-q|\mid p\neq q\in\Lambda_1\},\inf\{|p-q|\mid p\neq q\in\Lambda_2\},\inf\{|p-q|\mid p\neq q\in\Lambda\}\right),$$

and let $G_i(x)=\alpha_{i,p}\frac{x-p}{|x-p|^2}$ if $x\in B(p,\delta)$ for $p\in\Lambda_i$, and $G_i=0$ elsewhere.

Then, under the assumptions of the proposition, there exists $C>0$ such that for any $R>0$

$$\int_{B_R}|j_1-G_1|^2,\int_{B_R}|j_2-G_2|^2<CR^2.$$

Therefore

$$\int_{B_R} |j_1 + j_2 - (G_1 + G_2)|^2 < CR^2.$$

In view of the previous Lemma, Proposition 3 is proved. □

Corollary 2. *We have, with $m = 0$,*

$$\tilde{W}(\mathbb{Z}) < +\infty, \ \tilde{W}(\mathbb{N}) < +\infty$$

Proof. There exists $j_1 \in \mathcal{F}_{\mathbb{Z}^2}$ and from Corollary 1 there exists $j_2 \in \mathcal{F}_{\mathbb{Z}^2 \setminus \mathbb{Z}}$ such that $W(j_1)$ and $W(j_2)$ are both finite with $m = 1$. Then, by Proposition 3 and since $-\mathrm{div}(j_1 - j_2) = \sum_{p \in \mathbb{Z}} \delta_p$, and $\mathbb{Z}$ is uniform, we have $W(j_1 - j_2) < +\infty$ with $m = 0$, hence $\tilde{W}(\mathbb{Z}) < +\infty$. The proof for $\mathbb{N}$ is identical. □

Proposition 4. *For $m = 0$ we have*

$$W(\mathbb{Z}) < +\infty, \ W(\mathbb{N}) = +\infty$$

The case of $\mathbb{Z}$. We define $V_1(x) := -\log|\sin(\pi x)|$. Direct calculations lead to

$$-\triangle V_1 = 2\pi \sum_{p \in \mathbb{Z}} \delta_p \text{ in } \mathbb{R}^2$$

and

$$|\nabla V_1(x)| = \pi \frac{|\cos(\pi x)|}{|\sin(\pi x)|}.$$

Both $V_1(x)$ and $|\nabla V_1(x)|$ are 1-periodic functions. Straightforward calculations yield

$$W(\nabla V_1) < +\infty.$$

□

The case of $\mathbb{N}$. We must prove that no $\nabla U \in \mathcal{P}_{\mathbb{N}}$ is such that $W(\nabla U) < +\infty$. Our strategy is to construct $\nabla H_1 \in \mathcal{P}_{\mathbb{N}}$ such that $W(\nabla H_1) = +\infty$, and such that $W(\nabla H_1, \chi_R) < CR^2 \log^2 R$. Then, if there existed $\nabla H_2 \in \mathcal{P}_{\mathbb{N}}$ such that $W(\nabla H_2) < +\infty$, we would conclude that $W(\nabla(H_1 - H_2), \chi_R)$ grows at most like $R^2 \log^2 R$. Since $H_1 - H_2$ is harmonic we conclude from a Liouville type theorem that $\nabla(H_1 - H_2)$ is constant, which contradicts $W(\nabla H_1) = +\infty$.

To construct H_1 we use the Weierstass construction for a holomorphic function in the plane with a simple zeroe at each $p \in \mathbb{N}$ to define

$$H(x) := \Pi_{k \in \mathbb{N}} (1 - \frac{x}{k}) e^{\frac{x}{k}}.$$

Then we let

$$H_1(x) = -\log|H(x)|.$$

It is straightforward to check that the product in the definition of H converges uniformly on any compact subset of $\mathbb{C}$ and that

$$-\triangle H_1 = 2\pi \sum_{k \in \mathbb{N}} \delta_k \text{ in } \mathbb{R}^2$$

and for all $x \in \mathbb{C} = \mathbb{R}^2$

$$|H_1(x)| \leq \sum_{k \in \mathbb{N}} \left| \log(1 - \frac{x}{k}) + \frac{x}{k} \right| \tag{20}$$

and

$$|\nabla H_1(x)| = \left|\sum_{k\in\mathbb{N}} \frac{x}{k(k-x)}\right|. \tag{21}$$

Next, rather than proving $W(\nabla H_1, \chi_R) < CR^2 \log^2 R$, we prove the stronger, pointwise estimates:

$$|\nabla H_1(x)| \le C(\log(|x|+1)+1), \quad \text{outside } \cup_{k\in\mathbb{N}} B(k, \frac{1}{4}), \tag{22}$$

$$\left|\nabla H_1(x) + \frac{1}{x-k}\right| \le C(\log(|x|+1)+1), \quad \text{in } B(k, \frac{1}{4}). \tag{23}$$

For (22), take any $x \in \mathbb{C} \setminus \cup_{k\in\mathbb{N}} B(k, \frac{1}{4})$, it follows from (21) that

$$|\nabla H_1(x)| \le \sum_{1\le k\le[2|x|+1]} \left(\left|\frac{1}{k-x}\right| + \left|\frac{1}{k}\right|\right) + \sum_{k>[2|x|+1]} \left|\frac{x}{k(k-x)}\right| := I + II,$$

where $[\cdot]$ denotes the integer part of a real number. We have

$$II \le \sum_{k>[2|x|+1]} \frac{|x|}{(k-|x|)^2} \le |x| \int_{|x|}^{+\infty} \frac{dt}{t^2} \le 1,$$

$$\sum_{1\le k\le[2|x|+1]} \frac{1}{k} \le 1 + \int_1^{2|x|+1} \frac{dt}{t} \le 2(\log(|x|+1)+1).$$

On the other hand,

$$\sum_{1\le k\le[2|x|+1]} \left|\frac{1}{k-x}\right| \le \sum_{1\le k\le[2|x|+1]} \left|\frac{1}{\mathcal{R}e(k-x)}\right| \le 5 + 2\int_1^{2|x|+1} \frac{dt}{t} \le 5(\log(|x|+1)+1).$$

Therefore, for any $x \in \mathbb{C} \setminus \cup_{k\in\mathbb{N}} B(k, \frac{1}{4})$, we have $|\nabla H_1(x)| \le 8(\log(|x|+1)+1)$, and therefore (22) holds.

Now we prove (23). Let $x \in B(k, \frac{1}{4})$ for some $k \in \mathbb{N}$. As above

$$\left|\nabla H_1(x) + \frac{(\mathcal{R}e(x)-k, -\mathcal{I}m(x))}{|x-k|^2}\right| \le 8(\log(|x|+1)+1) + \frac{1}{k} \le 9(\log(|x|+1)+1),$$

or equivalently, if we use the division of complex number,

$$\left|\nabla H_1(x) + \frac{1}{x-k}\right| \le 8(\log(|x|+1)+1) + \frac{1}{k} \le 9(\log(|x|+1)+1),$$

since $x \in \mathbb{C} \setminus \cup_{i\ne k\in\mathbb{N}} B(i, \frac{1}{4})$. This proves (23)

We now turn to the proof that $W(\nabla H_1) = +\infty$. This is done by computing a lower bound for $|\nabla H_1(x)|$. More precisely we prove that or any $\varepsilon > 0$, there exists some positive constant C_1 depending on ε such that

$$|\nabla H_1(x)| \ge (\log(|x|+1) - C_1), \quad \text{if } |\mathcal{I}m(x)| \ge \varepsilon|x| + 1. \tag{24}$$

For this purpose we consider the meromorphic function

$$f(x) := \sum_{k\in\mathbb{N}} \frac{x}{k(k-x)}.$$

If $|\mathcal{I}m(x)| \geq \varepsilon|x|+1$, then $x \in \mathbb{C} \setminus \cup_{k\in\mathbb{N}} B(k,\frac{1}{4})$. Thus

$$\left| f(x) - \sum_{1\leq k\leq [2|x|+1]} \left(\frac{1}{k-x} - \frac{1}{k} \right) \right| \leq II \leq 1,$$

so that

$$\begin{aligned}\left| f(x) + \sum_{1\leq k\leq [2|x|+1]} \frac{1}{k} \right| &\leq 1 + \sum_{1\leq k\leq [2|x|+1]} \left| \frac{1}{k-x} \right| \\ &\leq 1 + \sum_{1\leq k\leq [2|x|+1]} \left| \frac{1}{\mathcal{I}m(x)} \right| \\ &\leq 1 + \sum_{1\leq k\leq [2|x|+1]} \frac{1}{|\mathcal{I}m(x)|} \\ &\leq 1 + \frac{2|x|+1}{|\mathcal{I}m(x)|} \\ &\leq 1 + 2/\varepsilon.\end{aligned}$$

On the other hand, we have

$$\sum_{1\leq k\leq [2|x|+1]} \frac{1}{k} \geq \log(|x|+1),$$

hence (24) follows. We claim that this implies that $W(\nabla H_1) = +\infty$.

To see this, we need to bound from below the integral of $\chi_R|\nabla H_1|^2$. We define g by applying Proposition 1 to ∇H_1 with $\delta = 1/4$. Then we deduce from (8), (9) and the fact that $\chi_R = 1$ on B_{R-1} that

$$\int \chi_R |\nabla H_1|^2 \geq \int_{B_{R-1}} |\nabla H_1|^2 - CR.$$

Then, integrating (24) on $\{x \in B_{R-1} \mid |\mathcal{I}m(x)| \geq \varepsilon|x|+1\}$ proves that $W(\nabla H_1) = +\infty$.

We may now argue by contradiction to prove the proposition. Assume that there exists $H_2 \in \mathcal{P}_{\mathbb{N}}$ such that $W(\nabla H_2) < +\infty$. Then $\bar{H} = H_2 - H_1$ is a harmonic function over $\mathbb{R}^2$. For $i = 1, 2$ we define g_i by applying Proposition 1 to ∇H_i with $\delta = 1/4$. Then

$$CR^2 \geq W(\nabla H_2, \chi_R) - W(\nabla H_1, \chi_R) \geq \int \chi_R(g_2 - g_1) - CR \geq \int_{B_{R-1}} (g_2 - g_1) - CR.$$

Then, letting $G(x) = (x-k)/|x-k|^2$ in $B(k, 1/4)$ for every k and $G = 0$ outside $\cup_k B(k, 1/4)$ we have, as in (19), for every k

$$\int_{B(k,1/4)} g_i = \int_{B(k,1/4)} \frac{1}{2}|\nabla H_i - G|^2 + C_0,$$

where $C_0 = -\pi \log 4$. Together with (9), this implies that

$$\int_{B_{R-1}} (g_2 - g_1) \geq \frac{1}{2} \int_{B_{R-1}\setminus \cup_k B(k,1/4)} \left(|\nabla H_2|^2 - |\nabla H_1|^2\right) - \frac{1}{2} \sum_{k=0}^{[R]} \int_{B(k,1/4)} \frac{1}{2}|\nabla H_1 - G|^2 - CR.$$

Using (23) we have

$$\int_{B(k,1/4)} \frac{1}{2}|\nabla H_1 - G|^2 \le C\left(\log(k+1)+1\right)^2,$$

so that

$$CR^2 \ge \frac{1}{2}\int_{B_{R-1}\setminus\cup_k B(k,1/4)} \left(|\nabla H_2|^2 - |\nabla H_1|^2\right) - CR\log^2 R.$$

Then, writing

$$|\nabla H_2|^2 - |\nabla H_1|^2 = |\nabla \bar H|^2 + 2\nabla\bar H\cdot\nabla H_1,$$

we find using (22) that on $B_{R-1}\setminus\cup_k B(k,1/4)$

$$|\nabla H_2|^2 - |\nabla H_1|^2 \ge |\nabla\bar H|^2 - C\log R|\nabla \bar H|,$$

and thus, letting $A_R = B_{R-1}\setminus\cup_k B(k,1/4)$,

$$CR^2 \ge \frac{1}{2}\int_{A_R}\left(|\nabla\bar H|^2 - C\log R|\nabla\bar H|\right) - CR\log^2 R,$$

from which we easily deduce

$$\int_{A_R} |\nabla \bar H|^2 \le CR^2\log^2 R.$$

It follows by a mean value argument that there exists $t\in[R/2, R-1]$ such that

$$\int_{\partial B_t} |\nabla\bar H|^2 \le CR\log^2 R,$$

and since $\bar H$ is harmonic, for any $x\in B_{R/4}$ we have

$$|\nabla^2\bar H(x)| \le \frac{1}{R^2}\int_{\partial B_t}|\nabla \bar H| \le C\frac{1}{R^2}\sqrt{R}R\log R.$$

Fixing x and letting $R\to\infty$, we find $\nabla^2\bar H(x) = 0$. Therefore $\nabla\bar H$ is a constant, which is clearly not possible since $W(\nabla H_1) = +\infty$ while $W(\nabla H_1 + \nabla\bar H) < +\infty$. □

We summarize the content of this section in the following

Theorem 1. *We have*

$$\tilde W(\mathbb{Z}) < +\infty,\ \tilde W(\mathbb{N}) < +\infty,\ \tilde W(\mathbb{Z}^2\setminus\mathbb{Z}) < +\infty,\ \tilde W(\mathbb{Z}^2\setminus\mathbb{N}) < +\infty \tag{25}$$

$$W(\mathbb{Z}) < +\infty,\ W(\mathbb{Z}^2) < +\infty,\ W(\mathbb{Z}^2\setminus\mathbb{Z}) < +\infty \tag{26}$$

$$W(\mathbb{N}) = +\infty,\ W(\mathbb{Z}^2\setminus\mathbb{N}) = +\infty \tag{27}$$

Proof. The result comes from Corollary 1, Corollary 2, Proposition 3 and Proposition 4. □

Remark 3. *The fact that $\tilde W(\mathbb{N}) < +\infty$ means that there exists $j\in\mathcal{F}_{\mathbb{N}}$ such that $W(j) < +\infty$. On the other hand, $\nabla U\in\mathcal{P}_{\mathbb{N}} \implies W(\nabla U) = +\infty$ since $W(\mathbb{N}) = +\infty$. One can then take the convolution of j with a regularizing function to find a vector field $\tilde j$ such that the average of $|\tilde j|^2$ over B_R is bounded independently of $\mathbb{R}$, while if $\triangle U = div\ \tilde j$, then $\limsup_{R\to+\infty} \fint_{B_R}|\nabla U|^2 = +\infty$.*

5. Sufficient conditions for finite renormalized energy

Theorem 2. *Given a discrete Λ, assume there exists $m \geq 0$ and $\varepsilon \in (0,1)$, $C > 0$ such that for any $x \in \mathbb{R}^2$ and for $R > 1$, we have*

$$\left|\sharp\left(B(x,R)\cap\Lambda\right) - m\pi R^2\right| \leq C R^{1-\varepsilon} \tag{28}$$

and

$$\inf_{x,y\in\Lambda, x\neq y} |x-y| > 0 \tag{29}$$

Then $W(\Lambda) < +\infty$, for the background m.

Remark 4. *For a Bravais lattice, the assumptions in the above theorem are satisfied. It was proved by Landau (1915) — see* [3] *for a more general statement — that the first assumption holds with $\varepsilon = 1/3$, see* [2] *for references on more recent developments.*

We recall a technical lemma.

Lemma 3. *(Theorem 8.17 in* [1]*) Assume $q > 2$ and $p > 1$ and v is a solution of the following equation*

$$-\triangle u = g + \sum_i \partial_i f_i$$

there exists some constant C such that

$$\|u\|_{L^\infty(B(0,R))} \leq C(R^{-\frac{2}{p}}\|u\|_{L^p(B(0,2R))} + R^{1-\frac{2}{q}}\|f\|_{L^q(B(0,2R))} + R^{2-\frac{4}{q}}\|g\|_{L^{q/2}(B(0,2R))})$$

Proof of Theorem 2. Assume Λ satisfies (28) and (29). The proof consists in constructing $j \in \mathcal{F}_\Lambda$ such that $W(j) < +\infty$, which is done by successive approximations constructing a first some U^1, then a correction U^2 to U^1, then a correction U^3 to $U^1 + U^2$, etc... In this construction, the U^k's are functions, and the sum of their gradients will converge to j.

Let $R_n = 2^{n-1}$. For all $p \in \Lambda$, we let U_p^1 be the solution to

$$\begin{cases} -\triangle U_p^1(y) & = & 2\pi\left(\delta_p(y) - \dfrac{\mathbf{1}_{B(p,R_1)}(y)}{\pi {R_1}^2}\right) & \text{in } B(p,R_1) \\ U_p^1(y) = \dfrac{\partial U_p^1}{\partial\nu}(y) & = & 0 & \text{on } \partial B(p,R_1) \end{cases}$$

where $\mathbf{1}_{B(x,r)}$ is the indicator function of the ball $B(x,r)$. The existence of a solution with Neumann boundary conditions follows from the fact that $\delta_p - \frac{\mathbf{1}_{B(p,R_1)}}{\pi {R_1}^2}$ has zero integral, and the radial symmetry of the solution implies U_p^1 is constant on the boundary, and the constant can be taken equal to zero. In fact, extending U_p^1 by zero outside $B(p,R_1)$, we get a solution of

$$-\triangle U_p^1(y) = 2\pi\left(\delta_p(y) - \frac{\mathbf{1}_{B(p,R_1)}(y)}{\pi R_1^2}\right)$$

in $\mathbb{R}^2$, which is supported in $B(p,R_1)$.

We let

$$U^1(y) := \sum_{p\in\Lambda} U_p^1(y).$$

This sum is well defined since, Λ being discrete, it is locally finite. Moreover U^1 solves

$$-\triangle U^1(y) = 2\pi\left(\sum_{p\in\Lambda}\delta_p - n_1(y)\right), \quad \text{where} \quad n_1(y) := \frac{\sharp\left(\Lambda\cap B(y,R_1)\right)}{\pi {R_1}^2} \tag{30}$$

Then we proceed by induction. For any $k\geq 2$ we let U^k be the solution to

$$\begin{cases} -\triangle U_p^k(y) &= 2\pi\left(\dfrac{\mathbf{1}_{B(p,R_{k-1})}(y)}{\pi {R_{k-1}}^2} - \dfrac{\mathbf{1}_{B(p,R_k)}(y)}{\pi {R_k}^2}\right) \quad \text{in } B(p,R_k) \\ U_p^k(y) = \frac{\partial U_p^k}{\partial\nu}(y) &= 0 \quad \text{on } \partial B(p,R_k), \end{cases} \tag{31}$$

and we let $U_p^k = 0$ outside the $B(p,R_k)$. We let $U^k(y) := \sum_{p\in\Lambda} U_p^k(y)$, so that

$$-\triangle U^k(y) = 2\pi\left(n_{k-1}(y) - n_k(y)\right),$$

where, for any $k\in\mathbb{N}$,

$$n_k(y) := \frac{\sharp\left(\Lambda\cap B(y,R_k)\right)}{\pi R_k^2}.$$

Now we study the convergence of $\sum_{k=1}^\infty \nabla U^k$.

First we note that there is an explicit formula for U_p^k. For any $k\geq 2$ we have

$$U_p^k(y) = V\left(\frac{y-p}{R_k}\right), \quad \text{where} \quad V(y) := \begin{cases} -\frac{3|y|^2}{2} + \ln 2 & \text{if } |y|\leq\frac12 \\ \frac{|y|^2}{2} - \ln|y| - \frac12 & \text{if } \frac12 < |y| \leq 1 \\ 0 & \text{if } |y|\geq 1, \end{cases}$$

from which it follows, since $\|\nabla U_p^k\|_\infty \leq \frac{C}{R_k}$ and the sum defining U^k has at most CR_k^2 non zero terms, that

$$\|\nabla U^k\|_\infty \leq CR_k. \tag{32}$$

Second we estimate $\|U^k\|_\infty$. We claim that

$$\forall k\geq 2,\ \exists C_k\in\mathbb{R} \text{ such that } \|U^k(y) - C_k\|_\infty = O(R_k^{1-\varepsilon}). \tag{33}$$

Indeed, from (28),

$$\|n_k - m\|_\infty \leq CR_k^{-1-\varepsilon}. \tag{34}$$

On the other hand, letting $n_y(r) := \sharp\left(B(y,r)\cap\Lambda\right)$, we have for any $y\notin\Lambda$

$$U^k(y) = \sum_{p\in B(y,R_k)\cap\Lambda} V\left(\frac{|p-y|}{R_k}\right) = \int_0^{R_k} V\left(\frac{t}{R_k}\right) n_y'(t)dt = -\int_0^{R_k}\frac{1}{R_k}V'\left(\frac{t}{R_k}\right)n_y(t)dt.$$

But, using (28), we have $n_y(t) = m\pi t^2 + O(t^{1-\varepsilon})$, hence

$$U^k(y) = -m\pi\int_0^{R_k}\frac{1}{R_k}V'\left(\frac{t}{R_k}\right)t^2dt + O(R_k^{1-\varepsilon}).$$

The first term is independent of y, we call it C_k. This proves (33).

Finally, we note that, from (34), it holds that

$$\|\triangle U^k\|_\infty = O(R_k^{-1-\varepsilon}). \tag{35}$$

Now, we claim that (33) and (35) imply that

$$\|\nabla U^k\|_\infty = O(R_k^{-\varepsilon}) \tag{36}$$

To see this we use the elliptic estimate of Lemma 3. For all $y \in \mathbb{R}^2$ we have

$$\int_{B(y,R_k)} |\nabla U^k|^2 = \int_{B(y,R_k)} |\nabla (U^k - C_k)|^2 \tag{37}$$

$$= -\int_{B(y,R_k)} \triangle U^k (U^k - C_k) + \int_{\partial B(y,R_k)} \frac{\partial U^k}{\partial \nu}(U^k - C_k) \le C{R_k}^2 \left(R_k^{-2\varepsilon} + \|\nabla U^k\|_\infty {R_k}^{-\varepsilon}\right)$$

Now we apply Lemma 3. For $i = 1, 2$ we have

$$\triangle \left(\partial_i U^k\right) = -2\pi \partial_i \left(n_{k-1} - n_k\right),$$

therefore for any $q > 2$ and $p > 1$,

$$\|\partial_i U^k\|_{L^\infty(B_{R_k/2})} \le C \left({R_k}^{-\frac{2}{p}} \|\partial_i U^k\|_{L^p(B_{R_k})} + {R_k}^{1-\frac{2}{q}} \|n_{k-1} - n_k\|_{L^q(B_{R_k})}\right).$$

Then, taking $p = 2$ and noting that (34) implies $\|n_{k-1} - n_k\|_q \le C{R_k}^{\frac{2}{q}-(1+\varepsilon)}$, we find using (37) that

$$\|\partial_i U^k\|_{L^\infty(B_{R_k/2})} \le C \left({R_k}^{-2\varepsilon} + {R_k}^{-\varepsilon} \|\nabla U^k\|_{L^\infty(B_{R_k})}\right)^{\frac{1}{2}} + C{R_k}^{-\varepsilon}.$$

This proves (36).

Now (36) implies that the sum $\sum_{k\ge 2} \nabla U^k$ converges, and if we let $j = \nabla U_1 + \sum_{k\ge 2} \nabla U^k$, then $-\mathrm{div} j = 2\pi(\sum_{p\in\Lambda} \delta_p - m)$, using (30), (31) and (34). Moreover j is a gradient since it is a sum of gradients, thus $j \in \mathcal{P}_\Lambda$.

To conclude, it is easy to check, using the assumption $\inf_{x,y\in\Lambda, x\ne y} |x-y| > 0$, that $W(\nabla U^1, \chi_R) \le CR^2$ for all $R > 1$, and to deduce using (36) that $W(j, \chi_R) \le CR^2$. □

For $\tilde{W}$ the hypothesis of Theorem 2 can be relaxed somewhat.

Theorem 2′. *Assume there exists some non-negative number $m \ge 0$ and some positive numbers $\varepsilon \in (0,1)$, $C > 0$ and a increasing sequence $\{R_n\}$ tending to $+\infty$ such that for any $x \in \mathbb{R}^2$ and for any $n \in \mathbb{N}$, we have*

$$\left|\sharp \left(B(x, R_n) \cap \Lambda\right) - m\pi R_n^2\right| \le C R_n^{1-\varepsilon},$$

and such that

$$\sum_n R_n^{-\varepsilon} < +\infty$$

and

$$\inf_{x,y\in\Lambda, x\ne y} |x-y| > 0.$$

Then $\tilde{W}(\Lambda) < +\infty$.

We will use the following simple estimate.

Lemma 4. *Let u be a solution of the following problem*

$$\begin{cases} -\triangle u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Then

$$\int_\Omega |\nabla u|^2 \leq C|\Omega|^2 \|f\|_\infty^2$$

where C is a constant independent of Ω.

Proof. We have

$$\int_\Omega |\nabla u|^2 = -\int_\Omega u\triangle u = \int_\Omega fu \leq \|u\|_1 \|f\|_\infty \leq \sqrt{|\Omega|}\|u\|_2\|f\|_\infty$$

Without loss of generality, we assume $\int u = 0$. By Poincaré inequality,

$$\|u\|_2 \leq C\sqrt{|\Omega|}\|\nabla u\|_2.$$

Finally, the desired result follows. □

Proof of Theorem $2'$. Let

$$\mu_\Lambda = \sum_{p\in\Lambda} \delta_p, \quad I_k = \frac{\mathbf{1}_{B_{R_k}}}{|B_{R_k}|},$$

for any integer $k > 0$, where $\mathbf{1}_{B_{R_k}}$ is the indicator function of the ball $B(0, R_k)$.

At the first step, for all $x \in \mathbb{R}^2$ we let U_x^1 be the solution to

$$\begin{cases} -\triangle U_x^1(y) &= 2\pi\left(\mu_\Lambda(y) - \mu_\Lambda * I_1(x)\right)\mathbf{1}_{B_{R_1}}(x-y) \quad \text{in } B(x, R_1+1) \\ \frac{\partial U_x^1}{\partial \nu}(y) &= 0 \quad \text{on } \partial B(x, R_1+1). \end{cases}$$

This equation has a solution which is unique up to an additive constant since

$$\int \left(\mu_\Lambda(y) - \mu_\Lambda * I_1(x)\right)\mathbf{1}_{B_{R_1}}(x-y)\,dy = \mu_\Lambda * (\pi {R_1}^2 I_1)(x) - \pi {R_1}^2 \mu_\Lambda * I_1(x) = 0.$$

We extend ∇U^1 by zero outside $B(x, R_1+1)$ and let

$$j^1(y) := \frac{1}{\pi {R_1}^2}\int_{\mathbb{R}^2} \nabla U_x^1(y)dx,$$

so that

$$-\mathrm{div}(j^1) = 2\pi\left(\sum_{p\in\Lambda}\delta_p - m_1(y)\right), \quad \text{where} \quad m_1 = \mu_\Lambda * I_1 * I_1.$$

Then we define j^k by induction. For $x \in \mathbb{R}^2$ we let U_x^k be the solution to

$$\begin{cases} -\triangle U_x^k(y) &= 2\pi\left(m_{k-1}(y) - m_{k-1} * I_k(x)\right)\mathbf{1}_{B_{R_k}}(x-y) \quad \text{in } B(x, R_k+1) \\ \frac{\partial U_x^k}{\partial \nu}(y) &= 0 \quad \text{on } \partial B(x, R_k+1), \end{cases}$$

and extend ∇U_x^k by 0 outside the ball $B(x, R_k)$. Then we let

$$j^k(y) := \frac{1}{\pi R_k^2}\int_{\mathbb{R}^2} \nabla U_x^k(y)dx$$

so that

$$-\mathrm{div}(j^k)(y) = 2\pi(m_{k-1}(y) - m_k(y)), \quad \text{where} \quad m_k = m_{k-1} * I_k * I_k.$$

We claim that

$$m_k(y) = m + O(R_k^{-1-\varepsilon}). \tag{38}$$

To see this, it suffices to note that from the commutativity of the convolution we have

$$m_k = (\mu_\Lambda * I_k) * (I_k * I_{k-1} * I_{k-1} * \cdots * I_1 * I_1).$$

Then from our first assumption $|\mu_\Lambda * I_k - m| \le C{R_k}^{-(1+\varepsilon)}$, which implies (38) since every I_k is a positive function with integral 1, and thus convoluting a function with it does not increase the L^∞ norm.

It follows from Lemmas 3 and 4 that for all $k \ge 2$ and $x \in \mathbb{R}^2$

$$\|\nabla U_x^k\|_{L^\infty(\mathbb{R}^2)} \le C{R_{k-1}}^{-\varepsilon},$$

which yields for all $k \ge 2$

$$\|j^k\|_{L^\infty(\mathbb{R}^2)} \le C{R_{k-1}}^{-\varepsilon}.$$

Therefore $\sum_{k\ge 2} \|j^k\|_\infty \le +\infty$ and we can define $j := \sum_{k\ge 1} j^k$. The vector field j solves

$$-\mathrm{div}(j) = 2\pi \left(\sum_{p\in\Lambda} \delta_p - m \right) \qquad \text{in } \mathbb{R}^2.$$

Now it suffices to prove that $W(j) < +\infty$. This is clearly a consequence of the fact that $W(j_1) < +\infty$ and the fact that $\sum_{k\ge 2} \|j^k\|_\infty \le +\infty$. On the other hand, $W(j_1) < +\infty$ is proved as follows: For any $p \in \Lambda$, and any $x \in B(p, R_1)$ we have $\|U_x^1(y) - log|y-p|\| < C$ in $C^1(B(p,\delta))$ with $C, \delta > 0$ independent of p, x and y, because of the equation satisfied by U^1 and the uniform spacing of the points in Λ. Also, if $x \notin B(p, R_1)$, then $\|U_x^1(y)\| < C$ in $C^1(B(p,\delta))$.

Then, since $j^1 = \int \nabla U_x^1 / \pi {R_1}^2$, we have $|j^1(y) - log|y-p|| < C$ in $B(p,\delta)$ for any $p \in \Lambda$ and $|j^1| < C$ outside $\cup_{p\in\Lambda} B(p,\delta)$. This implies that $W(j_1) < +\infty$. □

Proposition 5. *The conditions in Theorem 2 are optimal in some sense. More precisely, for any $m \ge 0$ and any $\varepsilon > 0$ there exists Λ such that $\tilde{W}(\Lambda) = +\infty$ and for any $x \in \mathbb{R}^2$ and any $R > 1$*

$$\left| \sharp \left(B(x,R) \cap \Lambda \right) - m\pi R^2 \right| \le CR^{1+\varepsilon}.$$

Proof. The counter-example is as follows, assuming without loss of generality that $\varepsilon < 1/2$: For all $k \in \mathbb{N}$, on the circle $\partial B(0, 4k)$, we distribute uniformly $[32\pi mk + k^\varepsilon]$ points, where $[x]$ is the integer part of x. This is clearly possible maintaining at the same time a distance greater than $\min(1/5m, 1)$ (if k is large enough) between the points, since $k^\varepsilon \ll k$ as $k \to +\infty$.

Then we have as $k \to +\infty$

$$\sharp \left(\Lambda \cap B(0,4k) \right) - m\pi(4k)^2 \simeq \sum_{i=1}^{k-1} [i^\varepsilon] \simeq \frac{k^{1+\varepsilon}}{1+\varepsilon},$$

thus for any j such that

$$-\mathrm{div}(j) = 2\pi \left(\sum_{p\in\Lambda} \delta_p - m \right),$$

and for any $R \in (4k+1, 4k+3)$, we have

$$\int_{\partial B(0,R)} j\cdot\nu = 2\pi \left(\sharp\left(\Lambda \cap B(0,R)\right) - m\pi R^2 \right) \simeq \frac{2\pi}{1+\varepsilon} k^{1+\varepsilon}.$$

Thus there exist $k_0 > 0$, $c_0 > 0$ such that if $k > k_0$ and for any $R \in (4k+1, 4k+3)$, we have

$$\frac{1}{2}\int_{\partial B(0,R)} |j|^2 \geq \frac{1}{4\pi R}\left(\int_{\partial B(0,R)} j\cdot\nu \right)^2 \geq c_0 k^{1+2\varepsilon} \tag{39}$$

Now we construct g using proposition 1 with $\delta < \frac{1}{2}\inf_{p\neq q\in\Lambda}|p-q|$ and $\delta < 1$. For functions $\{\chi_R\}_R$ satisfying (2), we have for any $k\in\mathbb{N}$ and since the support of χ_{4k+2} does not intersect $\cup_{p\in\Lambda} B(p,\delta)$ that

$$W(j,\chi_{4k+2}) = \int g\chi_{4k+2} = \int_{\cup_{p\in\Lambda}B(p,\delta)} g\chi_{4k+2} + \int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\delta)} g\chi_{4k+2}$$

and therefore, since $g = \frac{1}{2}|j|^2$ outside $\cup_{p\in\Lambda}B(p,\delta)$ and $g \geq -C$, we obtain

$$W(j,\chi_{4k+2}) \geq \sum_{i\leq k-1} \int_{B(0,4i+3)\setminus B(0,4i+1)} |j|^2 - CR^2 \geq CR^{2+2\varepsilon},$$

where we used (39) for the last inequality. Therefore $W(j) = +\infty$. □

6. Critical case

In view of Theorem 2 and Proposition 5, the critical discrepancy between $\sum_{p\in\Lambda}\delta_p$ and the uniform measure $m\,dx$ is when $\left|\sharp\left(B(x,R)\cap\Lambda\right) - m\pi R^2\right| = O(R)$. This includes the cases $\Lambda = \mathbb{Z}$ or $\mathbb{N}$. As shown by Theorem 1, we cannot expect $W(\Lambda)$ to be finite under such an assumption. However we have the following result for $\tilde{W}$.

Theorem 3. *Let $A \subset \mathbb{Z}^2$ and $\Lambda := \mathbb{Z}^2 \setminus A$. Assume there exists some constant $C > 0$ such that for all $x\in\mathbb{R}^2$ and for all $R > 1$ we have*

$$\sharp\left(A \cap B(x,R)\right) \leq CR.$$

Then, with $m = 1$,

$$\tilde{W}(\Lambda) < +\infty$$

This result is a direct consequence of Proposition 2 and the following:

Proposition 6. *Let $A \subset \mathbb{Z}^2$. Then the following properties are equivalent.*

Property I.: *There exists some constant $C > 0$ such that for all $x\in\mathbb{R}^2$ and for all $R > 1$ we have*

$$\sharp\left(A \cap B(x,R)\right) \leq CR \tag{40}$$

Property II.: *There exists a bijective map* $\Phi : \Lambda \to \mathbb{Z}^2$*, where* $\Lambda = \mathbb{Z}^2 \setminus A$*, satisfying*

$$\sup_{p\in\Lambda} |\Phi(p) - p| < \infty \tag{41}$$

The fact that the second property implies the first one is not difficult. First note that (41) is equivalent to the same property for Φ^{-1}, and that Property I is equivalent to the same property with squares K_R of sidelength R replacing the balls of radius R.

Now assume $\sharp(A \cap K_R) > CR$, then $\Phi^{-1}(K_R \cap \mathbb{Z}^2)$ is included in $\mathbb{Z}^2 \setminus A$ and thus contains at least CR points which do not belong to K_R. Therefore, as $C \to +\infty$, the maximal distance between an element p of $\Phi^{-1}(K_R \cap \mathbb{Z}^2)$ and K_R tends to $+\infty$. This proves that II $\implies$ I.

The proof of the converse is less obvious. It is essentially an application of the max-flow/min-cut duality, with arguments similar in spirit to those found in [10].

We let $\mathcal{G}$ be a graph for which the set of vertices is $\mathbb{Z}^2$ and the set of edges is

$$\mathcal{A} := \{(p, q) \quad | \quad p, q \in \mathbb{Z}^2, \|p - q\| = 1\}$$

where $\|\cdot\|$ is the Euclidean norm. Given an integer $N \in \mathbb{N}$, we define some function

$$\begin{array}{rccc} \mu_N : & \mathbb{Z}^2 & \to & \mathbb{R}_+ \\ & p & \mapsto & N^2 - \sharp(\Lambda \cap K_p^N) \end{array}$$

where $K_p^N := [kN, (k+1)N) \times [lN, (l+1)N)$ for $p = (k, l)$. Since $\Lambda = \mathbb{Z}^2 \setminus A$, μ_N is indeed non-negative and $\mu_N(p)$ is equal to $\sharp(A \cap K_p^N)$, i.e. the deficit of the points of Λ in K_p^N.

We introduce the following notions.

- A *flow*, or *1-form* is a map $\varphi : \mathcal{A} \to \mathbb{R}$ such that for any edge (p, q) one has $\varphi(p, q) = -\varphi(q, p)$.
- Given a flow φ, its divergence $\mathrm{div}(\varphi)$ is the function $\mathrm{div}(\varphi) : \mathbb{Z}^2 \to \mathbb{R}$ such that for any $p \in \mathbb{Z}^2$ one has
$$\mathrm{div}(\varphi)(p) := \sum_{(p,q)\in\mathcal{A}} \varphi(p, q)$$
- Given a function $f : \mathbb{Z}^2 \to \mathbb{R}$, its *gradient* ∇f is the 1-form $\nabla f(p, q) = f(q) - f(p)$.
- Given a subset A of $\mathbb{Z}^2$, its *boundary* ∂A is defined by
$$\partial A := \{(p, q) \in \mathcal{A} \mid p \in A, q \in \mathbb{Z}^2 \setminus A\}.$$
- Given a subset A of $\mathbb{Z}^2$, its *perimeter* is $\mathrm{Per}(A) := \sharp(\partial A)$
- A *curve* connecting p and q is a set of edges $\{(p_0, p_1), (p_1, p_2), \cdots, (p_{n-1}, p_n)\}$ with $p_0 = p$ and $p_n = q$. A *loop* or *cycle* is curve such that $p_n = p_0$. A graph is *connected* if any two points can be connected by a curve.
- Given a function $f : \mathbb{Z}^2 \to \mathbb{R}$ and $B \subset \mathbb{Z}^2$, its *integral* on B is defined by
$$\int_B f := \sum_{p\in B} f(p).$$
We denote also $f(B) = \int_B f$.
- Given a 1-form φ and a curve γ, the *integral* of φ on γ is defined by
$$\int_\gamma \varphi := \sum_{a\in\gamma} \varphi(a)$$

- Given two 1-forms φ and ϕ, their *inner product* is
$$\langle \varphi, \phi \rangle := \frac{1}{2} \sum_{a \in \mathcal{A}} \varphi(a)\phi(a)$$
- Given a 1-form φ and a subset $S \subset \mathcal{A}$, the *total variation* of φ with respect to S is defined by
$$[\varphi, S] := \frac{1}{2} \sum_{a \in S} |\varphi(a)|$$
When $S = \mathcal{A}$, we simply write $[\varphi]$.

We have the following classical results.

Lemma 5. *(Poincaré Lemma) Given a 1-form φ, if one has $\int_\gamma \varphi = 0$ for any loop γ, then there exists a function f satisfying*
$$\varphi = \nabla f$$

Proof. One fixes some point $p \in \mathbb{Z}^2$ and for any $q \in \mathbb{Z}^2$ one defines $f(q) := \int_{\mathcal{C}} \varphi$ where $\mathcal{C}$ is any curve connecting p and q. From the hypothesis, this definition is independent of the particular curve chosen, and it is easy to check that $\varphi = \nabla f$. □

Lemma 6. *(Stokes' formula) Let φ be a 1-form with compact support and f be a function with compact support. Then one has*
$$\langle \varphi, \nabla f \rangle = -\int_{\mathbb{Z}^2} f \, div(\varphi)$$

Proof. We write φ as linear combination of elementary 1-forms
$$\alpha_{(p,q)} := \delta_{(\{p,q)\}} - \delta_{\{(q,p)\}},$$
and note that
$$\langle \alpha_{(p,q)}, \nabla f \rangle = f(q) - f(p) = -\int_{\mathbb{Z}^2} f \mathrm{div}(\alpha_{(p,q)}),$$
since $\mathrm{div}(\alpha_{(p,q)}) = \delta_{\{p\}} - \delta_{\{q\}}$. □

Lemma 7. *(Coarea formula) Let $f : \mathbb{Z}^2 \to \mathbb{R}_+$ be a function with the compact support. Then one has*
$$[\nabla f] = \int_0^\infty \mathrm{Per}(\{f > t\}) dt$$

Proof. We note that
$$[\nabla f, \{(p,q),(q,p)\}] = |f(q) - f(p)|$$
and
$$\partial\{f > t\} \cap \{(p,q),(q,p)\} = \begin{cases} (p,q) & \text{if } f(p) > t \text{ and } f(q) \leq t \\ (q,p) & \text{if } f(q) > t \text{ and } f(p) \leq t \\ \emptyset & \text{otherwise,} \end{cases}$$
which implies that
$$\sharp(\partial\{f > t\} \cap \{(p,q),(q,p)\}) = \begin{cases} 1 & \text{if } f(p) > t \geq f(q) \text{ or } f(q) > t \geq f(p) \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we get

$$\int_0^\infty \sharp(\partial\{f>t\}\cap\{(p,q),(q,p)\})dt = |f(q)-f(p)|.$$

Summing with respect to all couples of edges $\{(p,q),(q,p)\}$ proves the result. □

We may now set up the duality argument. For any given 1-form φ we let

$$\|\varphi\|_\infty = \sup_{(p,q)\in\mathcal{A}} \varphi(p,q) = \sup\{\langle\phi,\varphi\rangle \mid \phi \text{ is compactly supported, } [\phi]\le 1\},$$

and define,

$$\alpha := \min_{-\mathrm{div}(\varphi)=\mu_N} \max\{\langle\phi,\varphi\rangle \mid \phi\in C_0, [\phi]\le 1\},$$

where C_0 is the set of compactly supported 1-forms.

Lemma 8. *One has*

$$\alpha = \max_{\nabla f\in C_0,[\nabla f]\le 1} \int_0^{+\infty}\left(\int_{\{f>t\}}\mu_N - \int_{\{f<-t\}}\mu_N\right)dt$$

Proof. By convex duality, we obtain

$$\alpha = \max_{\{\phi\in C_0|[\phi]\le 1\}} \min_{-\mathrm{div}(\varphi)=\mu_N} \langle\phi,\varphi\rangle. \tag{42}$$

Then given $\phi\in C_0$, we assume there exists a loop γ such that

$$\int_\gamma \phi \ne 0.$$

We may then define φ_t for any $t\in\mathbb{R}$ by

$$\varphi_t(a) := \begin{cases} t & \text{if } a\in\gamma \\ 0 & \text{otherwise}\end{cases}$$

Since γ is a loop, φ_t has compact support and $\mathrm{div}(\varphi_t)=0$. Moreover, since $\int_\gamma\phi\ne 0$,

$$\min_{t\in\mathbb{R}}\langle\phi,(\varphi+\varphi_t)\rangle = -\infty$$

which implies

$$\min_{-\mathrm{div}(\varphi)=\mu_N}\langle\phi,\varphi\rangle = -\infty.$$

As a consequence, the maximum in (42) can be restricted to those ϕ's for which the integral on any loop is zero, i.e. to gradients, in view of Lemma 5. Therefore

$$\alpha = \max_{\{\nabla f\in C_0|[\nabla f]\le 1\}} \min_{-\mathrm{div}(\varphi)=\mu_N} \langle\nabla f,\varphi\rangle \tag{43}$$

Now, from Lemma 6, we have

$$\alpha = \max_{\{\nabla f\in C_0|[\nabla f]\le 1\}} \min_{-\mathrm{div}(\varphi)=\mu_N} \int -\mathrm{div}(\varphi)f = \max_{\{\nabla f\in C_0|[\nabla f]\le 1\}} \int \mu_N f$$

On the other hand, for any function f with compact support we have as a well known consequence of Fubini's Theorem (see for instance [4], where this is named the bath-tub principle)

$$\int \mu_N f_+ = \int_0^{+\infty}\left(\int_{\{f>t\}}\mu_N\right)dt$$

and

$$\int \mu_N f_- = \int_0^{+\infty} \Big(\int_{\{f<-t\}} \mu_N \Big) dt.$$

Together with (43), this proves the result. □

Lemma 9. *Assuming Property I of Proposition 6, there exists $C > 0$ such that for any integer N and any finite $B \subset \mathbb{Z}^2$, we have*

$$\mu_N(B) \leq 4CN \operatorname{Per}(B)$$

Proof. Let $B_1, \cdots, B_k$ be the connected components of B. Then we have disjoint unions $B = \bigcup_{i=1}^k B_i$ and $\partial B = \bigcup_{i=1}^k \partial B_i$. Set $\tilde{B}_i := \bigcup_{p \in B_i} K_p^N$. We have $\mu_N(B_i) = \sharp \left(\tilde{B}_i \cap A \right)$, hence

$$\mu_N(B_i) \leq C \operatorname{diam}(\tilde{B}_i)$$

Now assume $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$ and $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)$ are in $\tilde{B}_i$ and such that $\operatorname{diam}(\tilde{B}_i) = \|\tilde{p} - \tilde{q}\|$. Without loss generality, we may assume that $\|\tilde{p} - \tilde{q}\| \leq 2(\tilde{p}_1 - \tilde{q}_1)$. There exists $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in B_i such that $\tilde{p} \in K_p^N$ and $\tilde{q} \in K_q^N$. Moreover,

$$\tilde{p}_1 - \tilde{q}_1 = N(p_1 - q_1) + (N-1) \leq N(p_1 - q_1 + 1).$$

On the other hand, from the connectedness of B_i, for any integer $x \in [p_1, q_1]$ we have $B_i \cap \{r\} \times \mathbb{Z} \neq \varnothing$ hence writing $m_x = \min\{y \mid (x,y) \in B_i\}$ and $M_x = \max\{y \mid (x,y) \in B_i\}$, the two edges $((x, m_x), (x, m_x - 1))$ and $((x, M_x), (x, M_x + 1))$ belong to ∂B_i. It follows that

$$\operatorname{Per}(B_i) = \sharp \partial B_i \geq 2(p_1 - q_1 + 1),$$

and then

$$\mu_N(B_i) \leq CN \operatorname{Per}(B_i), \quad \mu_N(B) = \sum_i \mu_N(B_i) \leq CN \sum_i \operatorname{Per}(B_i) = CN \operatorname{Per}(B).$$

□

As a consequence, we obtain

Corollary 3. *Assuming Property I of Proposition 6, there exists $C > 0$ and for any integer $N > 1$ there exists a 1-form φ such that*

$$-div(\varphi) = \mu_N \tag{44}$$

and for every edge $a \in \mathcal{A}$,

$$|\varphi(a)| \leq CN. \tag{45}$$

Proof. It follows from Lemmas 7 and 9 that

$$\int_0^{+\infty} \mu_N(\{f > t\}) \leq CN \int_0^{+\infty} \operatorname{Per}(\{f > t\}) = CN[\nabla f_+]$$

and

$$\int_0^{+\infty} \mu_N(\{f < -t\}) \leq CN \int_0^{+\infty} \operatorname{Per}(\{f > t\}) = CN[\nabla f_-].$$

This implies using Lemma 8 that

$$\alpha \leq CN \max_{\nabla f \in C_0, [\nabla f] \leq 1} [\nabla f] = CN.$$

Using the definition of α, there exists a 1-form φ with the desired properties (changing the constant to $2C$ for instance). $\square$

Proof of Proposition 6. We construct the bijective map $\Phi : \Lambda \to \mathbb{Z}^2$. This is done by specifying the for every $p, q \in \mathbb{Z}^2$ the number of points in $\Lambda \cap K_p^N$ whose images by Φ belong to $\mathbb{Z}^2 \cap K_q^N$, as follows:

$$n_{p\to q} := \begin{cases} \max(\varphi(p,q),0) & \text{if } (p,q)\in\mathcal{A} \\ \sharp(\Lambda\cap K_p^N) - \displaystyle\sum_{(p,q)\in\mathcal{A}} n_{p\to q} & \text{if } p=q \\ 0 & \text{otherwise,} \end{cases}$$

where φ is a flow satisfying (44), (45).

Now, for the numbers $n_{p\to q}$ to indeed correspond to a bijective map Φ we need to check some of their properties.

Property 1. If N is chosen large enough, then for any $p, q \in \mathbb{Z}^2$, we have $n_{p\to q} \geq 0$. This is clear when $p \neq q$. In the case $p = q$, we note that there are exactly 4 edges coming out of p. Thus, from (45) and the fact that $\sharp(\Lambda \cap K_p^N) \geq N^2 - CN$ we find (with another constant C still independent of N).

$$n_{p\to p} \geq N^2 - CN.$$

Thus we may indeed choose N large enough so that indeed $n_{p\to q} \geq 0$ for any $p, q \in \mathbb{Z}^2$.

Property 2. This one is clear from the definition of $n_{p\to q}$: For any $p \in \mathbb{Z}^2$ we have

$$\sum_q n_{p\to q} = \sharp(\Lambda \cap K_p^N).$$

Property 3. For any $q \in \mathbb{Z}^2$ we have

$$\sum_p n_{p\to q} = N^2.$$

Indeed, fixing $q \in \mathbb{Z}^2$ and all the sums below being with respect to p,

$$\begin{aligned} \sum_p n_{p\to q} &= n_{q\to q} + \sum_{(p,q)\in\mathcal{A}} n_{p\to q} \\ &= \sharp(\Lambda\cap K_q^N) - \sum_{(q,p)\in\mathcal{A}} n_{q\to p} + \sum_{(p,q)\in\mathcal{A}} n_{p\to q} \\ &= \sharp(\Lambda\cap K_q^N) + \sum_{(p,q)\in\mathcal{A},\varphi(p,q)\geq 0} \varphi(p,q) - \sum_{(q,p)\in\mathcal{A},\varphi(q,p)\geq 0} \varphi(q,p). \end{aligned}$$

Now since $\varphi(p,q) = -\varphi(q,p)$ we have

$$\sum_{(p,q)\in\mathcal{A},\varphi(p,q)\geq 0} \varphi(p,q) - \sum_{(q,p)\in\mathcal{A},\varphi(q,p)\geq 0} \varphi(q,p) = \sum_{(p,q)\in\mathcal{A}} \varphi(p,q) = -\mathrm{div}\varphi(q).$$

Using (44) this sum is equal to $\mu_N(q) = N^2 - \sharp(\Lambda \cap K_q^N)$, hence $\sum_p n_{p\to q} = N^2$.

The three properties insure that there exists a bijection $\Phi : \Lambda \to \mathbb{Z}^2$ such that for any $p, q \in \mathbb{Z}^2$ we have

$$n_{p\to q} = \sharp\{x \in \Lambda \cap K_p^N \mid \Phi(x) \in \mathbb{Z}^2 \cap K_q^N\}.$$

Since $n_{p\to q} \neq 0$ implies $\|p-q\| \leq 1$, we find that $\|\Phi(x) - x\| \leq 2\,\mathrm{diam}(K^N)$, for any $x \in \Lambda$. □

Remark 5. *The conclusion of Theorem 3 holds under the following, less restrictive assumption on Λ, which is assumed to be uniform, but not necessarily a subset of $\mathbb{Z}^2$:*

i) *There exists some positive constant $C > 0$ such that for any $x \in \mathbb{R}^2$ and any $R > 1$, one has $|\sharp(\Lambda \cap B(x,R)) - \pi R^2| \leq CR$.*

ii) *There exists some positive integer $N_0 \in \mathbb{N}$ such that for any $p \in \mathbb{Z}^2$, one has $\sharp\left(K_p^{N_0} \cap \Lambda\right) \leq N_0^2$.*

Indeed, the second assumption, implies that there exists an injective map

$$\Phi_p : K_p^{N_0} \cap \Lambda \to K_p^{N_0} \cap \mathbb{Z}^2.$$

We define $\Phi : \Lambda \to \mathbb{Z}^2$ to be the injective map whose restriction to $K_p^{N_0}$ is Φ_p for any $p \in \mathbb{Z}^2$ and let $\Lambda_1 = \Phi(\Lambda)$. Then Λ_1 is of the form $\mathbb{Z}^2 \setminus A$, with A satisfying (40). *Theorem 3 implies that $\tilde{W}(\Lambda_1) < +\infty$ and then from* (2) *we deduce that $\tilde{W}(\Lambda) < +\infty$.*

We conclude this section with

Theorem 3′. *Let $\Lambda \subset \mathbb{R}^2$ be discrete and uniform, and of the form $\Lambda = \Lambda_1 \times \mathbb{Z}$, where $\Lambda_1 \subset \mathbb{R}$.*

If there exists $C > 0$ such that for any $x \in \mathbb{R}^2$ and $R > 1$ we have $|\sharp(\Lambda \cap K(x,R)) - R^2| \leq CR$ — where $K(x,R)$ is the square with sidelength R and center x — then $\tilde{W}(\Lambda) < +\infty$.

Proof. The proof of the theorem will follow the same strategy as for Theorem 3, except that we work now in one dimension. For any integer $N > 0$ and $p \in \mathbb{Z}$ we let $I_p^N = [pN, (p+1)N)$ and $\mu^N(p) = N - \sharp(\Lambda_1 \cap I_p^N)$. We consider the graph with $\mathbb{Z}$ as the set of vertices and the set of edges

$$\mathcal{A} = \{(p,q) \mid p, q \in \mathbb{Z}, |p-q| = 1\}.$$

We claim that there exists $C > 0$, and for any integer $N > 0$ a 1-form $\varphi : \mathcal{A} \to \mathbb{R}$ such that

$$-\mathrm{div}(\varphi) = \mu_N, \quad \|\varphi\|_\infty \leq C. \tag{46}$$

Indeed we define φ as follows:

$$\varphi((k,k+1)) = \begin{cases} 0 & \text{if } k = 0 \\ -\displaystyle\sum_{i=1}^{k} \mu_N(i) & \text{if } k \geq 1 \\ \displaystyle\sum_{i=k+1}^{0} \mu_N(i) & \text{if } k < 0. \end{cases}$$

It is clear that $-\mathrm{div}(\varphi) = \mu_N$. Moreover, for instance if $k \geq 1$, then

$$\varphi((k,k+1)) = -\sum_{i=1}^{k} \left(N - \sharp(\Lambda_1 \cap I_p^N)\right) = \sharp(\Lambda_1 \cap [N,(k+1)N)) - kN.$$

But considering the square $K = [N,(k+1)N) \times [N,(k+1)N)$, we have

$$kN - \sharp\left(\Lambda_1 \cap [N,(k+1)N)\right) = \frac{1}{kN}\left((kN)^2 - \sharp(\Lambda \cap K)\right),$$

and thus using the hypothesis of the theorem we deduce that $|\varphi((k,k+1))| \le C$ as claimed.

Now we choose $N \ge 2C+1$ and following the proof of Proposition 6 we can construct a bijective map $\Phi : \Lambda_1 \to \mathbb{Z}^2$ such that $|\Phi(p) - p|$ is bounded independently of p. This induces a bijection with the same property from Λ to $\mathbb{Z}^2$, which proves Theorem 6, using Proposition 2. □

7. A Penrose lattice

We now describe the construction of a Penrose-type lattice Λ such that $\tilde{W}(\Lambda) < +\infty$. Of course it would be better to show that Λ satisfies the hypothesis of Theorem 2, but this to our knowledge an open problem.

For simplicity, we consider the Robinson triangle decompositions in Penrose's second tilling (P2)–kite and dart tiling, or in Penrose's third tilling (P3)–rhombus tiling, (see [8]). The construction is as follows: Ω_1 and Ω_2 are two Robinson triangles, namely, Ω_1 is an obtuse Robinson triangle having side lengths $(1,1,\varphi)$, while Ω_2 is an acute triangle with sidelengths $(\varphi,\varphi,1)$, where $\varphi = (1+\sqrt{5})/2$; the scaled-up domain $\varphi\Omega_1$ decomposes as the union of a copy of Ω_1 and a copy of Ω_2, where the interiors are disjoint — and such that $\varphi\Omega_2$ decomposes as the union of one copy of Ω_1 and two copies of Ω_2 with disjoint interiors (see figure).

For $i = 1,2$ we choose a point p_i in the interior of Ω_i.

Then we proceed by induction, starting with Ω_1 choosing p_1 as the origin, then scaling up by φ, then decomposing, then scaling up again, then decomposing each piece, etc... After n steps we have a (large domain) $\varphi^n\Omega_1$ tiled by a number of copies of either Ω_1 or Ω_2. In each tile we have a distinguished point, the union of these points is denoted Λ_n. As $n \to +\infty$ and modulo a subsequence, Λ_n converges to a discrete set Λ, which is uniform since the distance between two points is no less than $\min\left(d(p_1,\partial\Omega_1), d(p_2,\partial\Omega_2)\right)$.

Theorem 4. *We have $\tilde{W}(\Lambda) < +\infty$.*

Proof. For each n we define a current j_n as follows. On each copy of Ω_i we let j_n be equal to (a copy of) ∇U_i, where

$$\begin{cases} -\triangle U_i &= \delta_{p_i} - \frac{1}{|\Omega_i|} & \text{in } \Omega_i \\ \frac{\partial U_i}{\partial \nu} &= 0 & \text{on } \partial\Omega_i. \end{cases}$$

Then j_n converges as $n \to +\infty$ to a current j such that the following holds in $\mathbb{R}^2$

$$-\mathrm{div}(j) = \sum_{p\in\Lambda} \delta_p - \alpha,$$

where $\alpha = 1/|\Omega_i|$ on each copy of Ω_i. It is not difficult to check that $W(j) < +\infty$, but the background density α is not constant. We need to add a correction to j, which is the object of the following

Lemma 10. *There exist $m \in \mathbb{R}$ and a solution of the following equation in $\mathbb{R}^2$*

$$-div(j') = \alpha - m \tag{47}$$

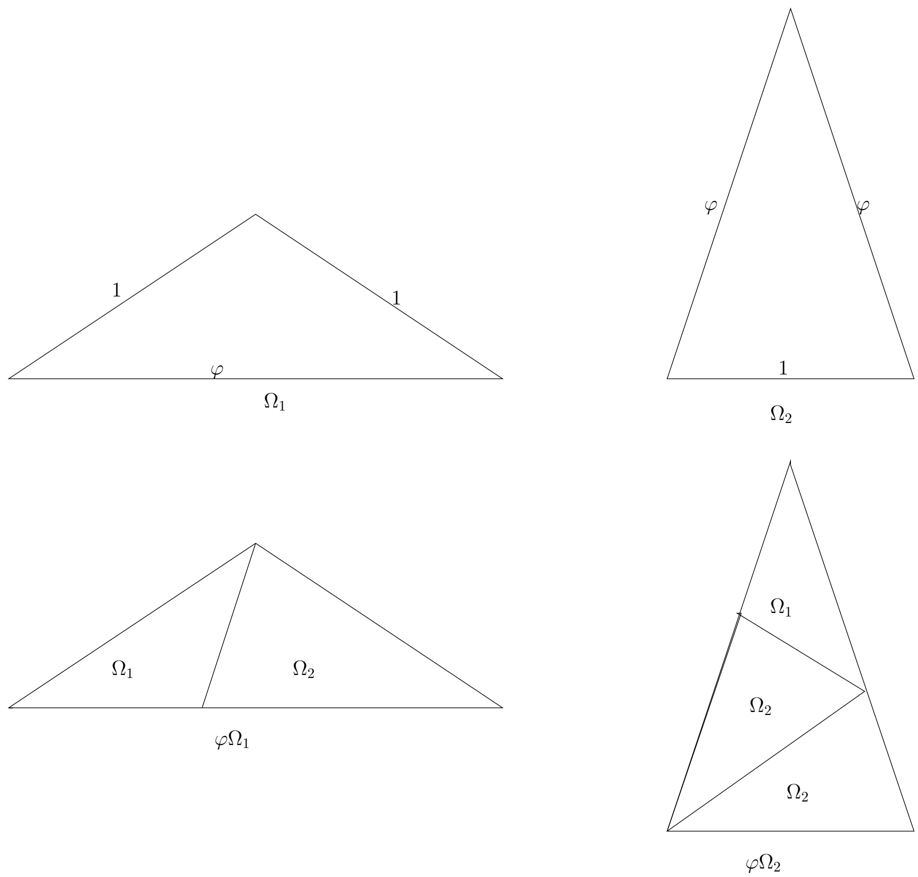

Figure 1

such that $\|j'\|_\infty < +\infty$,

Assuming the lemma is true we let $\tilde{j} = j + j'$. Then $-\mathrm{div}(\tilde{j}) = \sum_{p\in\Lambda} \delta_p - m$ thus $\tilde{j} \in \mathcal{F}_\Lambda$ for the background m, and the fact that $W(j) < +\infty$ and $j' \in L^\infty$ implies that $W(\tilde{j}) < +\infty$ and the Theorem. □

Proof of Lemma 10. The current j' is obtained as the limit of j_n, where j_n solves

$$\begin{cases} -\mathrm{div}(j_n) = \alpha_n - m_n & \text{in } \varphi^n\Omega_1 \\ j_n \cdot \nu = 0 & \text{on } \partial(\varphi^n\Omega_1), \end{cases} \tag{48}$$

where $\alpha_n : \varphi^n\Omega_1 \to \mathbb{R}$ is the function equal to $1/|\Omega_i|$ on each of the copies of Ω_i, $i = 1, 2$ which tile $\varphi^n\Omega_1$, and where m_n is equal to the average of α_n on $\varphi^n\Omega_1$.

The current j_n is defined recursively. First we define the equivalent of α_n for Ω_2-type domains: For any integer n we tile $\varphi^n\Omega_2$ by one copy of $\varphi^{n-1}\Omega_1$ and two copies of $\varphi^{n-1}\Omega_2$, then we tile each of the three pieces, etc... until we have tiled $\varphi^n\Omega_2$ by copies of either Ω_1 or Ω_2. then we let $\beta_n : \varphi^n\Omega_2 \to \mathbb{R}$ be the function equal to $1/|\Omega_i|$ on each of the copies of Ω_i, $i = 1, 2$. We also define q_n to be the equivalent of m_n, i.e. the average of β_n on $\varphi^n\Omega_2$.

Finally we define $\bar{j}_n$ to be the equivalent of j_n for type 2 domains, i.e. the solution of (48) with α_n replaced by β_n, m_n replaced by q_n and Ω_1 replaced by Ω_2.

Below it will be convenient to abuse notation by writing $\varphi^n\Omega_i$ for a copy of $\varphi^n\Omega_i$. Then we have $\varphi^n\Omega_1 = \varphi^{n-1}\Omega_1 \cup \varphi^{n-1}\Omega_2$. We let

$$j_n = j_{n-1}\mathbf{1}_{\varphi^{n-1}\Omega_1} + \bar{j}_{n-1}\mathbf{1}_{\varphi^{n-1}\Omega_2} + \nabla U_n \mathbf{1}_{\varphi^n\Omega_1}, \tag{49}$$

where

$$\begin{cases} -\triangle U_n &= (m_n - m_{n-1})\mathbf{1}_{\varphi^{n-1}\Omega_1} + (m_n - q_{n-1})\mathbf{1}_{\varphi^{n-1}\Omega_2} & \text{in } \varphi^n\Omega_1 \\ \frac{\partial U_n}{\partial \nu} &= 0 & \text{on } \partial(\varphi^n\Omega_1). \end{cases} \tag{50}$$

It is straightforward to check that j_n satisfies (48) assuming j_{n-1} and $\bar{j}_{n-1}$ do.

The relation (49) is the recursion relation which repeated n times allows to write j_n as equal to a sum of on the one hand error terms ∇U_k (or their type 2 equivalent that we denote V_k), for k between 1 and n, and on the other hand of a vector field which on each elementary tile of type Ω_1 of $\varphi^n\Omega_1$ is equal to j_0 and on a tile of type Ω_2 is equal to $\bar{j}_0$. However from (48) we may take $j_0 = 0$ and $\bar{j}_0 = 0$, thus we are left with evaluating the error terms.

Claim: There exists $C > 0$ such that for any integer $k > 0$ we have

$$\|\nabla U_k\|_\infty, \|\nabla V_k\|_\infty \le C\varphi^{-3k}.$$

This clearly proves that the sum of errors for $k = 1 \dots n$ is bounded in L^∞ independently of n and therefore that $\{j_n\}$ is bounded in L^∞. Then the limit j' is in L^∞.

To prove the lemma, it remains to prove the claim, and to show that j' satisfies (47) for some $m \in \mathbb{R}$, which in view of (48) amounts to showing that $\{m_n\}_n$ converges. For this we define u_{2n} (resp. u_{2n+1}) be the number of elementary tiles of type Ω_1 (resp. Ω_2) in $\varphi^n\Omega_1$. We define similarly v_{2n} and v_{2n+1} by replacing Ω_1 by Ω_2. Therefore $u_0 = 1$, $u_1 = 0$, $v_0 = 0$, $v_1 = 1$. We have the following recurrence relations

$$u_{2n+2} = u_{2n} + u_{2n+1}, \quad u_{2n+3} = u_{2n} + 2u_{2n+1},$$

which we can summarize as the single relation $u_{n+2} = u_{n+1} + u_n$. Similarly $v_{n+2} = v_{n+1} + v_n$. It follows that

$$u_n = \varphi^n \frac{1}{\varphi+2} + (-\varphi)^{-n}\frac{\varphi+1}{\varphi+2}, \quad v_n = \varphi^n \frac{\varphi}{\varphi+2} + (-\varphi)^{-n}\frac{-\varphi}{\varphi+2}.$$

We have $u_n = a\varphi^n + O(\varphi^{-n})$ and $v_n = b\varphi^n + O(\varphi^{-n})$ with $a = \frac{1}{\varphi+2}$ and $b = \frac{\varphi}{\varphi+2}$ strictly positive. Then we easily deduce that

$$m_n = \frac{u_{2n} + u_{2n+1}}{u_{2n}|\Omega_1| + u_{2n+1}|\Omega_2|} = m + O(\varphi^{-4n}),$$

where

$$m = \frac{1+\varphi}{|\Omega_1| + \varphi|\Omega_2|},$$

and similarly that $q_n = m + O(\varphi^{-4n})$. This proves in particular the convergence of $\{m_n\}_n$. Moreover it shows that the right-hand side of (50) is bounded by $C\varphi^{-4n}$. By elliptic regularity (lemma 3 and lemma 4) we deduce that

$$\|\nabla U_n\|_\infty \le C|\varphi^n\Omega_1|^{\frac{1}{2}}\varphi^{-4n} = C|\Omega_1|^{\frac{1}{2}}\varphi^{-3n},$$

and a similar bound for V_n. This proves the claim, and the lemma □

Remark 6. *The above construction could easily be generalized to similar recursive constructions.*

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