MONOMIALIZATION IN HIGHER DIMENSIONS

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Abstract

Suppose that K is an algebraically closed field of characteristic zero. A morphism $\Phi : K^n \rightarrow K^m$ is *monomial* if there exists $a_{ij} \in \mathbb{N}$ such that $y_i = \prod_{j=1}^{n} x_i^{a_{ij}}$ for $1 \leq i \leq m$, where $x_1, \ldots, x_n$ are the coordinates of K^n and $y_1, \ldots, y_m$ are the coordinates of K^m.

Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism of nonsingular K-varieties. Φ is *locally monomial* if for all $p \in X$, the germ of Φ at p is formally isomorphic to a germ of a monomial morphism (not necessarily at the origin). If global simple normal crossing divisors D_Y and $D_X = \Phi^{-1}(D_Y)$ exist which are compatible with these isomorphisms, then Φ is *toroidal.*

A *monomialization* (*toroidalization*) of Φ is a commutative diagram of morphisms

$$\begin{array}{rcl} X_1 & \stackrel{\Phi_1}{\rightarrow} & Y_1 \\ \Psi_1 \downarrow & & \downarrow \Psi_2 \\ X & \stackrel{\Phi}{\rightarrow} & Y \end{array}$$

such that Ψ_1, Ψ_2 are projective birational morphisms which are products of monoidal transforms (blow ups of nonsingular subvarieties) and Φ_1 is locally monomial (toroidal).

If Y is a curve, the construction of a monomialization (or toroidalization) is a consequence of resolution of singularities. However, new techniques are required when Y has higher dimension. There are several, relatively simple, proofs of monomialization and toroidalization when X and Y are surfaces. Unfortunately, these proofs do not readily extend to higher dimensions.

In previous work, the author has proven that a local monomialization can be constructed *locally along a fixed but arbitrary valuation* in arbitrary dimensions.

The author has also proven local monomialization and toroidalization when X is a 3-fold, and Y has arbitrary dimension (≤ 3).

In our local proof, invariants are used which measure the deviation of a morphism from being locally monomial, and behave well under blow ups. The invariants are tied to the rational rank of a particular valuation. The choice of centers is highly noncanonical, and is tied to the particular choice of valuation.

In our global proof in dimension 3, we found that natural invariants tend to behave very badly. For instance, the "kangaroo point" phenomenon can occur. This vexing problem also appears in resolution of singularities in characteristic p and in resolution of vector fields. At a kangaroo point, a natural invariant may go up by at most one after blowing up, but must eventually come down to at most the original value.

Recently, we have been able to globalize some of our local proof, above a fixed point of X such that $\Phi : X \rightarrow Y$ has been suitably prepared. Using this, we are able to prove local monomialization and toroidalization of dominant morphisms from a variety X of arbitrary dimension to a surface Y.