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The  $k$ -secant Lemma

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We report on a joint work with Laurent Gruson. Let  $X$  be a smooth variety imbedded in  $\mathbb{P}_N$ . For  $k = k_1 + \dots + k_r$ , we study the Hilbert-Boardman Scheme  $HB_{k_1, \dots, k_r}(X)$  of  $X$ . It parametrizes the finite linear subschemes  $Z \subset X$  supported in  $r$  points  $x_1, \dots, x_r$  (eventually coinciding) and such that  $k = d^0(Z)$  and  $d_{x_i}^0(Z) = k_i$ . We consider the incidence variety  $I_{HB} = \{(Z, y) \in HB_{k_1, \dots, k_r}(X) \times \mathbb{P}_N \text{ such that } Z \cup \{y\} \text{ is on a line}\}$ , and we prove the  $k$ -secant lemma: the general fiber of the second projection ( $I_{HB} \rightarrow \mathbb{P}_N$ ) is smooth of dimension  $N - 1 - kc + r$ , where  $c$  is the codimension of  $X$  in  $\mathbb{P}_N$  (the empty set has all dimensions).

As a special case (when  $r = 1$  and  $k = k_1$ ), we recover a celebrated theorem of Mather : higher polar varieties (for a general point) of a smooth variety are smooth.

Ziv Ran's famous  $(n + 2)$ -secant lemma can also be recovered as a consequence of our result.

As an obvious corollary of the  $k$ -secant lemma, we find that the general projection  $X_1$  of  $X$  to  $\mathbb{P}_{N-1}$  has generic singularities. In particular, the subvariety  $X_k \subset X_1$  formed by points of multiplicity  $\geq k$  has codimension  $k(c - 1)$  in  $\mathbb{P}_{N-1}$ , and is smooth outside  $X_{k+1}$ . To be more precise, the subvariety  $X_{\{k_1, \dots, k_r\}} \subset X_1$  formed by points whose fiber in  $X$  has support in  $r$  points  $\{x_1, \dots, x_r\}$  (eventually coinciding) and multiplicity  $\geq k_i$  at  $x_i$ , has codimension  $kc - r$  in  $\mathbb{P}_{N-1}$ , and is smooth outside  $X_{\{k_1, \dots, k_r, 1\}} = X_{\{k_1, \dots, k_r\}} \cap X_{k+1}$ .

As a consequence of this corollary, we prove a vanishing theorem for codimension 2 smooth varieties, that we conjectured in Trento twenty years ago:  $H^p(J_X(q)) = 0$  for  $p + q < N - 2$  and  $p < N - 1$ . Finally, with the help of Fyodor Zak, we conjecture a generalization of this result for smooth projective varieties of any codimension in a projective complex space, to be compared with Barth and Zak classical vanishing theorems.